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A Note on the Dirichlet-Neumann Problem in the Upper Half Unit Disc

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Research Article

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Abstract

This study addresses the solvability and explicit solutions of various boundary value problems (BVPs) for inhomogeneous equations within the upper half unit disc. Specifically, we investigate the Dirichlet, Neumann, and mixed Dirichlet-Neumann problems for the inhomogeneous Cauchy-Riemann and Bitsadze equations. By employing analytical techniques and function space theory, we establish necessary and sufficient conditions for the existence of solutions. Furthermore, explicit solution formulas are derived under these solvability criteria, providing a constructive approach to solving such BVPs. The significance of this research lies in its contribution to the broader theory of BVPs in complex domains. The results obtained not only extend classical boundary conditions but also offer a systematic framework for dealing with higher-order equations. The interplay between different boundary conditions is explored in detail, revealing new insights into the structure of solutions and their dependence on boundary data. Beyond the theoretical implications, our findings have potential applications in mathematical physics, fluid dynamics, and engineering, where such problems frequently arise in modeling physical phenomena. Future research may further extend these results to more general domains and nonlinear equations, enriching the field of complex analysis and partial differential equations.

1. Introduction

Boundary value problems (BVPs) play a significant role in mathematical analysis, particularly in the study of partial differential equations in complex domains. These problems frequently appear in diverse scientific and engineering fields, including fluid dynamics, electromagnetism, elasticity, and heat conduction. Among them, the Dirichlet and Neumann problems are of fundamental importance due to their wide range of applications and theoretical implications. Analyzing these problems in specific geometric domains, such as the upper half unit disc, provides valuable insights into solution behavior and the impact of boundary conditions.

One of the essential classes of PDEs studied in complex analysis involves the Cauchy-Riemann and Bitsadze equations. The solvability and explicit solutions of these equations under various boundary conditions have been extensively explored. Many researchers have contributed to the field by investigating different types of boundary conditions and their implications for

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PDEs. For instance, Darya and Tagizadeh (2024) analyzed the Dirichlet problem for the Cauchy-Riemann equations in the half-disc, while Chaudhary and Kumar (2009) examined BVPs in the upper half-plane. Additionally, Kumar and Prakash (2006) addressed mixed BVPs for inhomogeneous polyanalytic equations, highlighting the complexity introduced by such conditions.

In this study, we focus on the solvability conditions and explicit solutions of the Dirichlet, Neumann, and mixed Dirichlet-Neumann BVPs in the upper half unit disc. We employ analytical techniques, integral representations, and function space methods to derive necessary and sufficient conditions for the existence of solutions. This work builds upon and extends previous studies, incorporating a broader class of boundary conditions and addressing more generalized settings.

A central challenge in this study is the mixed Dirichlet-Neumann problem, which imposes different boundary conditions on different parts of the boundary. Understanding these types of problems is crucial for applications such as heat transfer, fluid-structure interactions, and stress analysis in elasticity. Previous works, including those of Karaca (2021, 2024a, 2024b) have demonstrated the significance of studying Schwarz-type, combined BVPs in complex domains, and explored Schwarz-type and combined BVPs, extending classical results in complex analysis. Our research further explores these topics by formulating and solving analogous problems in the upper half unit disc. Kalmenov et al. (2008) provided Green function representations for the Dirichlet problem of the polyharmonic equation in a sphere, contributing to the broader understanding of BVPs.

A major challenge in this research is the mixed Dirichlet-Neumann problem, which imposes different boundary conditions on different segments of the boundary. Understanding such problems is crucial for applications in heat transfer, fluid-structure interactions, and elasticity. Previous studies, including those of Begehr et al. (2008), Wang and Du (2004), Begehr and Vaitekhovich (2008 and 2012), Begehr et al. (2017) and Karachik (2013 and 2019) have highlighted the significance of Dirichlet and combined BVPs in complex domains. Our research expands upon these topics by formulating and solving analogous problems in the upper half unit disc.

To develop the theoretical foundation for our results, we first introduce key mathematical tools, including the complex forms of the Gauss divergence theorem and the Cauchy-Pompeiu representation formula. These fundamental results provide the basis for establishing the solvability conditions for the Dirichlet and Neumann problems.

This paper is organized as follows: In Section 1, we provide the necessary theoretical background, including the formulation of the inhomogeneous Cauchy-Riemann and Bitsadze equations. Section 2 is devoted to the Dirichlet and Neumann problems, where we establish the solvability criteria and construct explicit solutions. Finally, Section 3 presents concluding remarks and potential directions for future research, including extensions to higher-order equations and applications in applied mathematics and engineering.

By systematically analyzing these BVPs, we aim to contribute to the broader understanding of PDEs in complex domains and provide a solid foundation for further investigations into related mathematical models.

A complex-valued function $\omega = u + iv$ given by two real-valued functions u and v of the real variables x and y will be denoted by $\omega(z)$ although being rather a function of z and $\bar{z}$. In case when ω is independent of $\bar{z}$ in an open set of the complex plane $\mathbb{C}$ it is an analytic function. It then satisfies the Cauchy-Riemann system of first order partial differential equations

$$u_x = v_y, \qquad u_y = -v_x.$$

This is equivalent to

$$\omega_{\bar{z}} = 0$$

as follows from

$$2\partial_{\bar{z}}\,\omega = (\partial_x + i\partial_y)(\,u + iv) = \partial_x u - \partial_y v + i(\partial_x v + \partial_y u).$$

Using these complex derivatives, the classical Gauss divergence theorem can be rewritten in a complex form for functions that are continuously differentiable in a bounded domain D with a smooth boundary ∂D. The key results that follow from this representation include the Gauss theorem (G.T.) and the Cauchy-Pompeiu representation (C.-P. r.), which are fundamental in deriving the solvability conditions for our BVPs.

Gauss Theorem (Complex Form) (G.T.) Let $D \subset \mathbb{C}$ be a regular domain (i.e. a bounded domain with smooth boundary) and let $\omega \in C^1(D;\,\mathbb{C}) \cap C(\,\bar{D}\,;\mathbb{C})$. Then

$$\int_D \omega_{\bar{z}}\,(z)dxdy = \frac{1}{2i}\int_D \omega\,(z)\,dz$$

and

$$\int_D \omega_{z}\,(z)dxdy = -\frac{1}{2i}\int_D \omega\,(z)\,d\bar{z}$$

for $z = x + iy$.

Cauchy–Pompeiu representation (C.–P. r.) Let D and ω be as above. Then

$$\omega(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{\omega(\varsigma)}{\varsigma - z}\,d\varsigma - \frac{1}{\pi}\int_D \frac{\omega_{\bar{\varsigma}}(\varsigma)}{\varsigma - z}\,d\xi d\acute{\eta}$$

and

$$\omega(z) = -\frac{1}{2\pi i}\int_{\partial D}\frac{\omega(\varsigma)}{\overline{\varsigma - z}}\,d\bar{\varsigma} - \frac{1}{\pi}\int_D \frac{\omega_{\varsigma}(\varsigma)}{\overline{\varsigma - z}}\,d\xi d\acute{\eta}$$

for $\varsigma = \xi + i\acute{\eta}$ and $z \in D$.

These results serve as the foundation for establishing the existence and uniqueness of solutions to the Dirichlet and Neumann problems.

The following theorem presents the necessary and sufficient conditions for the solvability of the Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the upper half unit disc.

Theorem 1. (Darya and Tagizadeh, 2024) The Dirichlet boundary value problem for the inhomogeneous Cauchy-Riemann equation

$\omega_{\bar{z}} = g(z)$ in $\mathbb{D}^+$, $\omega = \Upsilon_0$ on $\partial\mathbb{D}^+$ $g \in L_p(\mathbb{D}^+;\mathbb{C})$, $p > 2$, $\Upsilon_0 \in L_2(\mathbb{R},\mathbb{C}) \cap C(\partial\mathbb{D}^+,\mathbb{C})$

is solvable if and only if for $z \in \mathbb{D}^+$,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\Upsilon_0(\varsigma)\left[\frac{1}{\varsigma - \bar{z}} + \frac{\bar{z}}{\varsigma\bar{z} - 1}\right]d\varsigma - \frac{1}{\pi}\int_{\mathbb{D}^+} g(\varsigma)\left[\frac{1}{\varsigma - \bar{z}} + \frac{\bar{z}}{\varsigma\bar{z} - 1}\right]d\xi d\acute{\eta} = 0 \quad (1)$$

and its solution can be uniquely expressed as

$$\omega(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \Upsilon_0(\varsigma)\left[\frac{1}{\varsigma - z} + \frac{z}{\varsigma z - 1}\right] d\varsigma - \frac{1}{\pi}\int_{\mathbb{D}^+} g(\varsigma)\left[\frac{1}{\varsigma - z} + \frac{z}{\varsigma z - 1}\right] d\xi d\acute{\eta}, \tag{2}$$

where $\varsigma = \xi + i\acute{\eta}$.

The following theorem presents the necessary and sufficient conditions for the solvability of the Neumann problem for the inhomogeneous Cauchy-Riemann equation in the upper half unit disc.

Theorem 2. (Karaca, 2024b) The Neumann problem for the inhomogeneous Cauchy- Riemann equation in the upper half unit disc

$\omega_{\bar{z}}$ =g(z) in $\mathbb{D}^+$, $\partial_\nu\omega = \Upsilon$ on $\partial\mathbb{D}^+$, $\omega(0) = c$

for $g \in L_p(\mathbb{D}^+;\mathbb{C}), p > 2,\ \Upsilon \in L_2(\mathbb{R},\mathbb{C}) \cap C(\partial\mathbb{D}^+,\mathbb{C}), c \in \mathbb{C}$

is solvable if and only if for $z \in \mathbb{D}^+$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \left(\Upsilon(\varsigma) - \bar{\varsigma}\, g(\varsigma)\right)\left[\frac{-1}{\varsigma - \bar{z}} + \frac{1}{\varsigma(1-\varsigma\bar{z})}\right] d\varsigma \\ + \frac{1}{\pi}\int_{\mathbb{D}^+} g(\varsigma)\left[\frac{\bar{z}}{(\varsigma - \bar{z})^2} + \frac{\bar{z}}{(1-\varsigma\bar{z})^2}\right] d\xi d\acute{\eta} = 0.\end{aligned} \tag{3}$$

The unique solution then is

$$\begin{aligned}\omega(z) = c - \frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \left(\Upsilon(\varsigma) - \bar{\varsigma}\, g(\varsigma)\right)\left[\log\left(\frac{\varsigma - z}{\varsigma}\right) - \log(1-\varsigma z)\right]\frac{d\varsigma}{\varsigma} - \frac{1}{\pi}\int_{\mathbb{D}^+} g(\varsigma)\left[\frac{z}{\varsigma(\varsigma - z)}\right. \\ \left. + \frac{z}{1-\varsigma z}\right] d\xi d\acute{\eta}.\end{aligned} \tag{4}$$

By the help of Theorem 2, we have the following theorem, which is a special form of Neumann problem.

Theorem 3. (Karaca, 2024b) The problem

$\omega_{\bar{z}}$ =g(z) in $\mathbb{D}^+$, $z\,\omega_z = \Upsilon$ on $\partial\mathbb{D}^+$, $\omega(0) = c$

for $g \in L_p(\mathbb{D}^+;\mathbb{C}), p > 2,\ \Upsilon \in L_2(\mathbb{R},\mathbb{C}) \cap C(\partial\mathbb{D}^+,\mathbb{C}), c \in \mathbb{C}$

is solvable if and only if for $z \in \mathbb{D}^+$,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \Upsilon(\varsigma)\left[\frac{-1}{\varsigma - \bar{z}} + \frac{1}{\varsigma(1-\varsigma\bar{z})}\right] d\varsigma + \frac{1}{\pi}\int_{\mathbb{D}^+} g(\varsigma)\left[\frac{\bar{z}}{(\varsigma - \bar{z})^2} + \frac{\bar{z}}{(1-\varsigma\bar{z})^2}\right] d\xi d\acute{\eta} = 0. \tag{5}$$

The unique solution then is

$$\omega(z) = c - \frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \Upsilon(\varsigma)\left[\log\left(\frac{\varsigma - z}{\varsigma}\right) - \log(1-\varsigma z)\right]\frac{d\varsigma}{\varsigma} - \frac{1}{\pi}\int_{\mathbb{D}^+} g(\varsigma)\left[\frac{z}{\varsigma(\varsigma - z)} + \frac{z}{1-\varsigma z}\right] d\xi d\acute{\eta}. \tag{6}$$

2. The Dirichlet-Neumann Problem

In this section, we establish the theoretical foundation necessary for analyzing the BVPs presented in later sections. We begin by reviewing the inhomogeneous Cauchy-Riemann and Bitsadze equations within the upper half unit disc, outlining their significance and mathematical properties. This groundwork is essential for formulating the solvability conditions and deriving explicit solutions.

The subsequent results build upon previously known theorems, extending them to encompass more complex boundary conditions. By leveraging analytical techniques and employing function spaces suited to the problem's geometry, we derive conditions that guarantee the existence and uniqueness of solutions.

Theorems 4 and 5, presented below, address the Dirichlet-Neumann problem and a more general boundary value problem for the inhomogeneous Bitsadze equation, respectively. These results highlight the interplay between boundary conditions and the inhomogeneous nature of the governing equations, providing a robust framework for further analysis.

Theorem 4. The Dirichlet-Neumann problem for the inhomogeneous Bitsadze equation in the upper half unit disc

$\omega_{\bar{z}\bar{z}} = \mathrm{g}(z)$ in $\mathbb{D}^+, \omega = \Upsilon_0,\ \partial_\nu \omega_{\bar{z}} = \Upsilon_1$ on $\partial\mathbb{D}^+, \quad \omega_{\bar{z}}(0) = \mathrm{c},$

for $\mathrm{g} \in \mathrm{L}_{\mathrm{p}}(\mathbb{D}^+;\mathbb{C}), \mathrm{p} > 2,\ \Upsilon_0, \Upsilon_1 \in \mathrm{L}_2(\mathbb{R},\mathbb{C}) \cap C(\partial\mathbb{D}^+,\mathbb{C}), \mathrm{c} \in \mathbb{C}$ is solvable if and only if for $z \in \mathbb{D}^+$

$$c + \frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \Upsilon_0(\varsigma)\left[\frac{1}{\bar{z}(\varsigma-\bar{z})} + \frac{1}{\varsigma\bar{z}-1}\right] d\varsigma + \frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \left(\Upsilon_1(\varsigma) - \bar{\varsigma}\, g(\varsigma)\right) \frac{1-|z|^2}{|z|^2}\left[log\left(\frac{\varsigma - z}{\varsigma}\right) - log(1-\varsigma z)\right]\frac{d\varsigma}{\varsigma} + \frac{1}{\pi}\int_{\mathbb{D}^+} g(\varsigma)(|z|^2 - |t|^2)\left[\frac{1}{\bar{z}\varsigma(z-\varsigma)} + \frac{1}{\bar{z}(z\varsigma-1)}\right] d\xi d\acute{\eta} = 0 \tag{7}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \left(\Upsilon_1(\varsigma) - \bar{\varsigma}\, g(\varsigma)\right)\left[\frac{-1}{\varsigma-\bar{z}} + \frac{1}{\varsigma(1-\varsigma\bar{z})}\right] d\varsigma + \frac{1}{\pi}\int_{\mathbb{D}^+} g(\varsigma)\left[\frac{\bar{z}}{(\varsigma-\bar{z})^2} + \frac{\bar{z}}{(1-\varsigma\bar{z})^2}\right] d\xi d\acute{\eta} = 0. \tag{8}$$

The solution then is

$$\omega(z) = c\bar{z} + \frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \Upsilon_0(\varsigma)\left[\frac{1}{\varsigma - z} + \frac{z}{\varsigma z - 1}\right] d\varsigma + \frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \left(\Upsilon_1(\varsigma) - \bar{\varsigma}\, g(\varsigma)\right)\frac{1-|z|^2}{z}\left[log\left(\frac{\varsigma - z}{\varsigma}\right) - log(1-\varsigma z)\right]\frac{d\varsigma}{\varsigma} + \frac{1}{\pi}\int_{\mathbb{D}^+} g(\varsigma)(|\varsigma|^2 - |z|^2)\left[\frac{1}{\varsigma(\varsigma - z)} + \frac{1}{1-\varsigma z}\right] d\xi d\acute{\eta}. \tag{9}$$

Proof. The problem is equivalent to the system

$\omega_{\bar{z}} = \mathrm{u}$ in $\mathbb{D}^+$, $\omega = \Upsilon_0$ on $\partial\mathbb{D}^+$,

$u_{\bar{z}} = \mathrm{g}(z)$ in $\mathbb{D}^+$, $\partial_\nu \mathrm{u} = \Upsilon_1$ on $\partial\mathbb{D}^+$, $\mathrm{u}(0) = \mathrm{c}$.

The solvability conditions are

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \Upsilon_0(\varsigma)\left[\frac{1}{\varsigma-\bar{z}} + \frac{\bar{z}}{\varsigma\bar{z}-1}\right] d\varsigma - \frac{1}{\pi}\int_{\mathbb{D}^+} u(\varsigma)\left[\frac{1}{\varsigma-\bar{z}} + \frac{\bar{z}}{\varsigma\bar{z}-1}\right] d\xi d\acute{\eta} = 0 \tag{10}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\left(\Upsilon_1(\varsigma)-\overline{\varsigma}\,g(\varsigma)\right)\left[\frac{-1}{\varsigma-\bar{z}}+\frac{1}{\varsigma(1-\varsigma\bar{z})}\right]d\varsigma+\frac{1}{\pi}\int_{\mathbb{D}^+}g(\varsigma)\left[\frac{\bar{z}}{(\varsigma-\bar{z})^2}+\frac{\bar{z}}{(1-\varsigma\bar{z})^2}\right]d\xi d\acute{\eta} = 0 \tag{11}$$

and the unique solutions are

$$\omega(z)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\Upsilon_0(\varsigma)\left[\frac{1}{\varsigma-\mathrm{z}}+\frac{\mathrm{z}}{\varsigma\mathrm{z}-1}\right]d\varsigma-\frac{1}{\pi}\int_{\mathbb{D}^+}u(\varsigma)\left[\frac{1}{\varsigma-\mathrm{z}}+\frac{\mathrm{z}}{\varsigma\mathrm{z}-1}\right]d\xi d\acute{\eta} \tag{12}$$

and

$$u(z)=c-\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\left(\Upsilon_1(\varsigma)-\overline{\varsigma}\,g(\varsigma)\right)\left[log\left(\frac{\varsigma-z}{\varsigma}\right)-log(1-\varsigma z)\right]\frac{d\varsigma}{\varsigma}-\frac{1}{\pi}\int_{\mathbb{D}^+}g(\varsigma)\left[\frac{z}{\varsigma(\varsigma-z)}+\frac{\mathrm{z}}{1-\varsigma z}\right]d\xi d\acute{\eta} \tag{13}$$

according to Theorems 1 and 2. Substituting the Eq. (13) into the Eqs. (10) and (12), we get the desired result.

Theorem 5. The boundary value problem for the inhomogeneous Bitsadze equation in the upper half unit disc

$\omega_{\bar{z}\bar{z}}=g(z)\ in\ \mathbb{D}^+,\ \omega=\Upsilon_0,\ z\omega_{z\bar{z}}=\Upsilon_1$ on $\partial\mathbb{D}^+,\ \omega_{\bar{z}}(0)=\mathrm{c}$

is solvable for $\mathrm{g}\in\mathrm{L}_1(\mathbb{D}^+;\mathbb{C}),\ \Upsilon_0,\Upsilon_1\in C(\partial\mathbb{D}^+,\mathbb{C}),\ \mathrm{c}\in\mathbb{C}$ if and only if for $z\in\mathbb{D}^+$,

$$\begin{aligned}c&+\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\Upsilon_0(\varsigma)\left[\frac{1}{\bar{z}(\varsigma-\bar{z})}+\frac{1}{\varsigma\bar{z}-1}\right]d\varsigma\\&+\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\left(\Upsilon_1(\varsigma)-\overline{\varsigma}\,g(\varsigma)\right)\frac{1-|z|^2}{|z|^2}\left[log\left(\frac{\varsigma-z}{\varsigma}\right)-log(1-\varsigma z)\right]\frac{d\varsigma}{\varsigma}\\&+\frac{1}{\pi}\int_{\mathbb{D}^+}g(\varsigma)(|z|^2-|t|^2)\left[\frac{1}{\bar{z}\varsigma(z-\varsigma)}+\frac{1}{\bar{z}(z\varsigma-1)}\right]d\xi d\acute{\eta}=0\end{aligned} \tag{14}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\Upsilon_1(\varsigma)\left[\frac{-1}{\varsigma-\bar{z}}+\frac{1}{\varsigma(1-\varsigma\bar{z})}\right]d\varsigma+\frac{1}{\pi}\int_{\mathbb{D}^+}g(\varsigma)\left[\frac{\bar{z}}{(\varsigma-\bar{z})^2}+\frac{\bar{z}}{(1-\varsigma\bar{z})^2}\right]d\xi d\acute{\eta}=0 \tag{15}$$

holds. The solution then is uniquely given by

$$\begin{aligned}\omega(z)=c\bar{z}&+\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\Upsilon_0(\varsigma)\left[\frac{1}{\varsigma-\mathrm{z}}+\frac{\mathrm{z}}{\varsigma\mathrm{z}-1}\right]d\varsigma\\&+\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\Upsilon_1(\varsigma)\frac{1-|z|^2}{z}\left[log\left(\frac{\varsigma-z}{\varsigma}\right)-log(1-\varsigma z)\right]\frac{d\varsigma}{\varsigma}\\&+\frac{1}{\pi}\int_{\mathbb{D}^+}g(\varsigma)(|\varsigma|^2-|z|^2)\left[\frac{1}{\varsigma(\varsigma-z)}+\frac{1}{1-\varsigma z}\right]d\xi d\acute{\eta}.\end{aligned} \tag{16}$$

Proof. The proof follows the same steps as Theorem 4, but with Theorem 3 used in place of Theorem 2.

3. Conclusion

In this study, we analyzed the solvability and explicit solutions of the Dirichlet, Neumann, and Dirichlet-Neumann BVPs for inhomogeneous equations in the upper half unit disc. By leveraging fundamental techniques from complex analysis and functional spaces, we established necessary and sufficient conditions for the existence of solutions. Our results provide a systematic framework for solving these problems and extend previously known results by incorporating mixed boundary conditions.

A key contribution of this work is the explicit construction of solutions for the Dirichlet-Neumann problem, which highlights the interplay between different types of boundary conditions. By expressing solutions in a closed form, we offer a constructive approach that can be directly applied in further mathematical and applied studies. These findings not only deepen our theoretical understanding of BVPs but also have potential applications in mathematical physics, fluid dynamics, and engineering problems where such equations naturally arise.

Moreover, the methodology presented in this study can be extended to more general settings, including higher-order equations and different geometric configurations. Future research directions could explore nonlinear extensions of these problems, the impact of additional boundary constraints, and numerical methods for approximating solutions in cases where explicit formulas are difficult to obtain.

In conclusion, this study contributes to the broader literature on BVPs in complex domains by providing a rigorous analytical framework for inhomogeneous equations in the upper half unit disc. The results presented here not only unify and extend existing theories but also pave the way for new developments in applied mathematics and theoretical physics.

Authorship Contribution Statement

The author is solely responsible for the conceptualization, methodology, analysis, and manuscript preparation.

Conflict of Interest

The author declares no conflict of interest.

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