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A Novel Semi-Analytical Approach for High-Order Delay Differential Equations Based on History Functions: Application to Nonlinear Vibration of Delayed Systems

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Abstract

In this study, a novel semi-analytical approach based on the Perturbation-Iteration Algorithm is proposed for solving high-order delayed differential equations using history functions. By employing the method of steps to transform the delayed problem into a system of ordinary differential equations defined over sub-intervals, the proposed approach offers a systematic solution framework distinct from existing methods in the literature. Another significant contribution of this study is the development of an algorithmic procedure for determining the initial function that initiates the iteration process. By incorporating the history function and continuity conditions between consecutive intervals directly into the governing equations through matrix operations, this procedure enables the algorithm to generate smooth and high-precision solutions within each sub-interval. The proposed method is applied to the dynamic analysis of the delayed Mathieu and delayed damped Mathieu equations, which play a critical role in nonlinear vibration theory. Parametric investigations reveal that positive delay coefficients effectively suppress vibration amplitudes by introducing an artificial damping effect, while negative delay coefficients may counteract physical damping and drive the system toward instability; additionally, increases in excitation amplitude intensify oscillatory responses and slightly reduce convergence speed. The results demonstrate that the developed approach is an effective and reliable tool for modelling and analyzing complex engineering problems involving delay terms.

1. Introduction

Delay differential equations (DDEs) are essential mathematical tools for modelling dynamic processes in which the rate of change of a system depends not only on its current state but also on its past states. As described by Polyanin et al. (2023) and Ford (2025), phenomena involving "after-effect" and time delay, encountered in fields such as engineering, biology (Rihan, 2021), and economics, can fundamentally alter a system's dynamic behavior. Accurate modelling of delay terms is crucial for system stability in high-precision industrial applications, such as predicting chatter vibrations in milling processes (Yang et al., 2023), as also demonstrated in high-dimensional delay differential equation studies of crossflow-induced nonlinear vibrations in steam generator tubes (Sun et al., 2023).

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Obtaining analytical solutions for DDEs is inherently more challenging than for classical ordinary differential equations. This has led researchers to develop efficient numerical and semi-analytical methods. The literature presents various numerical techniques, including Variable Multistage Methods (VMM) for higher-order DDEs (Martín & García, 2002) and the Direct Adams-Moulton Method (DAM) for second-order problems (Seong et al., 2013). Kumar and Methi (2021) proposed efficient numerical algorithms for nonlinear DDEs, while Khuri and Sayfy (2017) developed a Green's function-based iterative approach for functional differential equations.

In addition to numerical methods, semi-analytical methods are often preferred for solving complex problems. Barde and Maan (2019) proposed the Natural Homotopy Analysis Method for nonlinear DDEs, Olvera et al. (2015) used the Enhanced Multistage Homotopy Perturbation Method (EMHPM), and Aziz and Amin (2016) used Haar wavelets to provide approximate solutions for delayed equations. Amirali (2023) established stability inequalities for linear nonhomogeneous Volterra delay integro-differential equations. Recently, Ohira and Ohira (2025) introduced new approaches involving the Fourier transform for solving DDEs.

In recent years, polynomial-based matrix methods have become prominent in solving complex differential equations due to their high accuracy and wide applicability. Çevik et al. (2025) provided a comprehensive review of polynomial matrix collocation methods in engineering applications. Additionally, Cayan et al. (2022) proposed effective solutions for linear and nonlinear engineering models using a Taylor-splitting collocation approach. In this context, Dağ and Biçer (2026) developed a new method for nonlinear DDEs using Boole polynomials. Previous work has also addressed the solution of pantograph-type delayed equations using orthoexponential polynomials (Bahşı et al., 2015), Adomian decomposition method (Alenazy, et al., 2022), Taylor polynomials (Yüzbaşı & Ismailov, 2018) and neural networks (Bahşı & Bahşı, 2025).

The Mathieu equation, a central focus of this study, plays a key role in the stability analysis of parametrically excited systems. In their study on generalizations of the Mathieu equation, Kovacic et al. (2018) highlighted the importance of stability charts for these equations. However, the analysis of Mathieu equations with time delay and damping still requires novel, high-precision solution methods, particularly due to the need to account for history functions (Blanco-Cocom et al., 2012).

The Perturbation-Iteration Algorithm (PIA) has been successfully applied to solve pantograph-type (variable delay qt) differential equations (Bahşi & Çevik, 2015). However, while the delay in pantograph-type equations is proportional, in many engineering problems the delay is a constant duration τ $(t-\tau)$, and the system behaviour is determined by a history function $h(t)$. In such constant delay problems, it is necessary to divide the solution domain into sub-intervals and ensure continuity within each interval.

In this study, unlike existing methods in the literature, a novel semi-analytical algorithm based on the PIA is presented for high-order delay differential equations using history functions. The main innovation of the proposed method is the integration of the PIA with the "method of steps," which divides the solution domain into sub-intervals, and the inclusion of a systematic matrix-based procedure for determining initial functions that ensures continuity at each sub-interval transition. This approach aims to provide faster convergence and greater accuracy than methods currently available in the literature. Unlike existing step-based and semi-analytical methods that impose inter-interval continuity in a limited manner, the proposed approach introduces a systematic matrix-based continuity enforcement that simultaneously satisfies all derivative matching conditions, resulting in smoother, more robust, and faster-converging solutions for high-order delay differential equations.

The study is structured as follows: Section 2 outlines the mathematical framework of the method. Section 3 evaluates the accuracy of the proposed method by comparing it with numerical methods from the literature. Section 4 applies the method to the dynamic analysis of the delayed Mathieu equation (DME) and delayed damped Mathieu equation (DDM), which are important in computational mechanics, to examine the effects of delay and damping parameters on system stability.

2. Methodology

This section presents the mathematical framework for solving delay differential equations using the PIA. The solution strategy is based on the "method of steps," which transforms the delayed problem into a sequence of ordinary differential equations defined over sub-intervals. First, the general formulation of the problem and the domain decomposition strategy are introduced. Next, the derivation of the PIA(1,1) algorithm is detailed, followed by the procedure for determining the initial functions required to ensure continuity between successive intervals.

2.1. General formulation of delay differential equations based on history functions

The general form of the k-th order delay differential equation, which includes the delay term $u(t-\tau)$ or at least one of its derivatives is given in implicit form as:

$$F\left(u^{(k)},\dots,u',u,u_\tau^{(k)},\dots,u_\tau',u_\tau,\varepsilon,t\right)=0 \tag{1}$$

where $0\le t\le b$, $u=u(t)$, $u_\tau=u(t-\tau)$ such that $\tau>0$, and ε is the perturbation parameter. The conditions for this equation, where $\alpha_{1i},\alpha_{2i},\alpha_{3i}\in R$ for all i, $\beta\in R$ such that $a<\beta<b$, and $h(t)$ is defined in the interval $[-\tau,0]$, are given as:

$$\sum_{j=0}^{k-1}\left(\alpha_{1i}u^{(j)}(0)+\alpha_{2i}u^{(j)}(\beta)+\alpha_{3i}u^{(j)}(b)\right)=\gamma_i,\quad i=1,\dots,k-1 \tag{2}$$

and the history function is defined as:

$$u(t)=h(t),\quad -\tau\le t\le 0 \tag{3}$$

In the solution of Eq. (1) for the first sub-interval $[0,\tau]$, the delay term $u(t-\tau)$ is calculated using the history function $h(t)$. Specifically, for $t\in[0,\tau]$, the relationship is defined as $u(t-\tau)=h(t-\tau)$. Within this same sub-interval, any derivative of the delay term corresponds to the derivative of the history function of the same order. By substituting the history function or its derivatives into the corresponding delay terms, the solution over the interval $[0,\tau]$ is obtained using the proposed PIA algorithm. For the remaining sub-intervals of the domain $[0,b]$ where $t\ \ge\tau$, the history function is defined by the PIA solution obtained in the preceding sub-interval. Consequently, the global solution for the entire domain is constructed by combining the solutions derived from all sub-intervals.

2.2. Domain decomposition

The domain of the problem $[0,b]$ is divided into J sub-intervals as follows:

$$J=\left\lceil\frac{b-0}{\tau}\right\rceil \tag{4}$$

where $\lceil.\rceil$ denotes the ceiling function. The domain $D=[0,b]$ is expressed as the union of sub-intervals D_j for $j=1,2,\dots,J$:

$$D = \bigcup_{j=1}^{J} D_j = \left(\bigcup_{j=1}^{J-1} [(j-1)\tau, j\tau] \right) \cup [(J-1)\tau, b] \tag{5}$$

The dependent variable $u(t)$ is defined as u_{D_j} in the sub-interval D_j. The equation is converted into a system of equations for each sub-interval:

$$F^{D_j}\left((u^{D_j})^{(k)}, \dots, (u^{D_j})', u^{D_j}, \varepsilon, t\right) = 0, \quad t \in D_j \tag{6}$$

For $j > 1$, the delay terms in the interval D_j are equal to the n-th iteration solution of the previous interval $u_n^{D_{j-1}}(t)$. The global n-th iteration solution $u_n(t)$ is defined as:

$$u_n(t) = \begin{cases} u_n^{D_1}(t) & , \quad t \in D_1 \\ u_n^{D_2}(t) & , \quad t \in D_2 \\ \vdots & \vdots \quad \vdots \\ u_n^{D_J}(t) & , \quad t \in D_J \end{cases} \quad (7) \tag{7}$$

2.3. PIA(1,1) algorithm

For the n-th iteration solution $u_n^{D_j}$ in the interval D_j, a direct expansion with a single correction term is applied:

$$(u^{D_j})_{n+1} = (u^{D_j})_n + \varepsilon \left(u_c^{D_j}\right)_n, \qquad j = 1, 2, \dots, J \tag{8}$$

where $\left(u_c^{D_j}\right)_n$ is the n-th correction term. Substituting Eq. (8) into Eq. (6) and expanding in a Taylor series up to the first order derivative yields:

$$\begin{aligned} & F^{D_j}\left(({u^{D_j})_n}^{(k)}, \dots, (u^{D_j})_n, 0, t\right) \\ & \quad + F^{D_j}_{\left(u^{D_j}\right)^{(k)}}\left(({u^{D_j})_n}^{(k)}, \dots, (u^{D_j})_n, 0, t\right) \varepsilon \left(\left(u_c^{D_j}\right)_n\right)^{(k)} \\ & \quad + F^{D_j}_{\left(u^{D_j}\right)'}\left(({u^{D_j})_n}^{(k)}, \dots, (u^{D_j})_n, 0, t\right) \varepsilon \left(\left(u_c^{D_j}\right)_n\right)' \\ & \quad + F^{D_j}_{u^{D_j}}\left(({u^{D_j})_n}^{(k)}, \dots, (u^{D_j})_n, 0, t\right) \varepsilon (u_c)_n^{D_j} \\ & \quad + F^{D_j}_{\varepsilon}\left(({u^{D_j})_n}^{(k)}, \dots, (u^{D_j})_n, 0, t\right) \varepsilon = 0 \end{aligned} \tag{9}$$

This equation can be simplified as the PIA(1,1) algorithm:

$$F^{D_j}_{\left(u^{D_j}\right)^{(k)}} \left(u_c^{D_j}\right)_n^{(k)} + \cdots + F^{D_j}_{\left(u^{D_j}\right)'} \left(u_c^{D_j}\right)_n' + F^{D_j}_{u^{D_j}} \left(u_c^{D_j}\right)_n = -F_\varepsilon^{D_j} - \frac{F^{D_j}}{\varepsilon} \tag{10}$$

2.4. Determination of initial functions

For $j > 1$, the initial function $u_0^{D_j}(t)$ for the interval D_j is determined using the PIA(1,1) solution obtained in the previous interval D_{j-1}, ensuring continuity at the boundary $(j-1)$. The continuity conditions are:

$$\left(u_0^{D_j}\right)^{(m)} \left((j-1)\tau\right) = \left(u_n^{D_{j-1}}\right)^{(m)} \left((j-1)\tau\right), \quad m = 0, 1, \dots, k-1 \tag{11}$$

The initial function is defined by a polynomial of degree $(k-1)$, $P_{k-1}^{D_j}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{k-1} t^{k-1}$. The coefficient matrix $\mathbf{A}^{D_j}$ of this polynomial is obtained by:

$$\mathbf{A}^{D_j} = [a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_{k-1}]^T = (\mathbf{M}^{D_j})^{-1}\mathbf{B}^{D_j} \tag{12}$$

Here, $\mathbf{M}^{D_j}$ and $\mathbf{B}^{D_j}$ are defined as follows:

$$\mathbf{M}^{D_j} = \begin{bmatrix} 1 & (j-1)\tau & ((j-1)\tau)^2 & \cdots & ((j-1)\tau)^{k-1} \\ 0 & 1 & 2(j-1)\tau & \cdots & (k-1)((j-1)\tau)^{k-2} \\ 0 & 0 & 2 & & (k-1)(k-2)((j-1)\tau)^{k-3} \\ & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (k-1)! \end{bmatrix}$$

$$\mathbf{B}^{D_j} = \begin{bmatrix} u_n^{D_{j-1}}((j-1)\tau) \\ \left(u_n^{D_{j-1}}\right)'((j-1)\tau) \\ \left(u_n^{D_{j-1}}\right)''((j-1)\tau) \\ \vdots \\ \left(u_n^{D_{j-1}}\right)^{(k-1)}((j-1)\tau) \end{bmatrix}$$

3. Numerical Validation

In this section, the proposed PIA(1,1) algorithm is applied to a benchmark delay differential equation to validate its accuracy and efficiency. The obtained semi-analytical results are compared with the exact solution and other numerical methods available in the literature to demonstrate the performance of the proposed approach.

Consider the second-order linear delay differential equation given by:

$$u''(t) + u(t) = u(t-1), \quad t \in [0,2] \tag{13}$$

subject to the history function and initial condition:

$$u(t) = t^2 + 3t + 2, \quad -1 \le t \le 0; \quad u'(0) = 0$$

According to the domain decomposition strategy described in Section 2, since the delay is $\tau = 1$ and the domain is $b = 2$, the problem is divided into $J = 2$ sub-intervals: $D_1 = [0,1]$ and $D_2 = [1,2]$. The global solution is constructed as:

$$u_n(t) = \begin{cases} u_n^{D_1}(t) & , \quad t \in D_1 \\ u_n^{D_2}(t) & , \quad t \in D_2 \end{cases}$$

Solution for the First Interval ($\mathbf{D_1}$)

For $t \in [0,1]$, the delay term $u(t-1)$ falls into the history interval $[-1,0]$. Thus, the perturbation equation is constructed using the known history function $h(t)$:

$$(u^{D_1})'' + \varepsilon u^{D_1} - h(t-1) = 0$$

Substituting $h(t-1) = (t-1)^2 + 3(t-1) + 2$ and applying the PIA(1,1) expansion, the correction term $\left(u_c^{D_1}\right)$ is derived. To determine the initial function $u_0^{D_1}(t)$, the boundary conditions at $t = 0$ are used. The coefficient matrix yields a simple starting function $u_0^{D_1}(t) = = P_1^{D_1}(t) = 2$. Following the iteration process, the solution for the first interval is obtained as:

$$u_1^{D_1}(t) = \frac{t^4}{12} + \frac{t^3}{6} - t^2 + 2$$

$$u_2^{D_1}(t) = -\frac{t^6}{360} - \frac{t^5}{120} + \frac{t^4}{6} + \frac{t^3}{6} - \frac{t^2}{2} + 2$$

Solution for the Second Interval ($\mathbf{D_2}$)

For $t \in [1,2]$, the delay term $u(t-1)$ depends on the solution $u^{D_1}(t)$ found in the previous step. The system equation becomes:

$$(u^{D_2})'' + \varepsilon u^{D_2} - u_n^{D_1}(t-1) = 0$$

Crucially, to ensure continuity at the node point $t = 1$, the initial function $u_0^{D_2}(t)$ is determined using the continuity conditions:

$$u_0^{D_2}(1) = u_n^{D_1}(1), \quad \left(u_0^{D_2}\right)'(1) = \left(u_n^{D_1}\right)'(1)$$

Using the matrix operation defined in Eq. (12), the coefficient matrix $\mathbf{A}_{D_2}$ is calculated, yielding the initial function $u_0^{D_2}(t) = \frac{797}{360} - \frac{107}{120}t$. By employing this initial function in the PIA algorithm, the polynomial solution for the second interval is computed as follow:

$$u_1^{D_2}(t) = \frac{t^8}{20160} + \frac{t^7}{5040} - \frac{t^6}{180} - \frac{t^5}{120} + \frac{t^4}{6} + \frac{t^3}{6} - t^2 + 2$$

$$u_2^{D_2}(t) = -\frac{t^{10}}{1814400} - \frac{t^9}{362880} + \frac{t^8}{10080} + \frac{t^7}{5040} - \frac{t^6}{360} - \frac{t^5}{120} + \frac{t^4}{6} + \frac{t^3}{6} - t^2 + 2$$

Finally, combining the results from both steps, the global solution at the second iteration is expressed as:

$$u_2(t) = \begin{cases} -\frac{t^6}{360} - \frac{t^5}{120} + \frac{t^4}{6} + \frac{t^3}{6} - t^2 + 2 & , \quad t \in D_1 \\ -\frac{t^{10}}{1814400} - \frac{t^9}{362880} + \frac{t^8}{10080} + \frac{t^7}{5040} & \\ -\frac{t^6}{360} - \frac{t^5}{120} + \frac{t^4}{6} + \frac{t^3}{6} - t^2 + 2 & , \quad t \in D_2 \end{cases}$$

The results obtained by the PIA(1,1) algorithm are compared with the VMM (Martín & García, 2002) and the-DAM (Seong et al., 2013). Figure 1 presents the absolute errors at selected points.

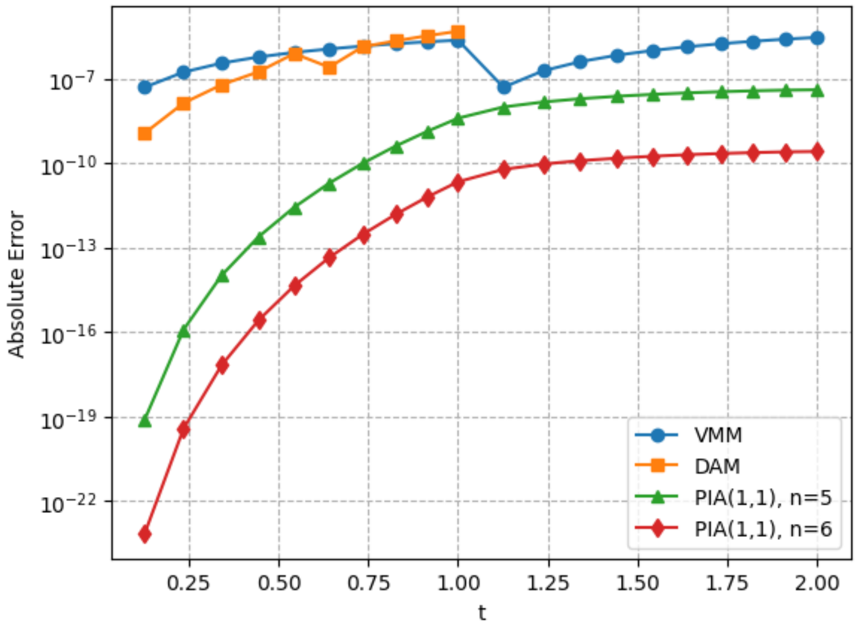

Figure 1. Comparison of absolute errors for PIA(1, 1) against VMM and DAM methods

As demonstrated in Figure 1, the PIA(1,1) algorithm achieves significantly lower error rates (10^{-16} to 10^{-24} range) compared to VMM and DAM (10^{-6} to 10^{-9} range). Furthermore, regarding computational cost defined by the "Totally Called Function" (FCN) count, PIA requires only 52 evaluations for $n = 5$ and 60 for $n = 6$, whereas DAM requires 61-73 evaluations. Here, FCN denotes the total number of function evaluations required by the algorithm and is used as a hardware-independent measure of computational cost. The maximum absolute error decreases exponentially with the number of iterations, achieving high precision in minimal CPU time (approx. 7.66s for $n = 6$).

The convergence speed is further analyzed in Table 1. It is observed that while the CPU time increases linearly with the number of iterations, the error decreases exponentially, achieving a maximum absolute error of 2.61×10^{-10} in just 7.66 seconds.

Table 1. Maximum absolute error and CPU time for different iterations

n	2	3	4	5	6
E_n	$1.25e-02$	$3.10e-04$	$4.44e-06$	$4.05e-08$	$2.61e-10$
CPU time	$2.69s$	$3.94s$	$5.26s$	$6.48s$	$7.66s$

4. Applications in Computational Mechanics

In this section, the efficacy of the proposed PIA is demonstrated by applying it to complex engineering problems modeled by delay differential equations. Specifically, the method is employed to analyze the dynamic behavior of the DME and the DDME, which appear frequently in nonlinear vibration theory and control systems.

4.1. Dynamic analysis of the DME

The Mathieu equation is a quintessential model in the study of parametrically excited systems, describing phenomena such as the vibration of elliptical membranes and the stability of structures under periodic loading. A classic physical realization of this problem is a mathematical pendulum with a vertically moving support, as illustrated in Figure 2.

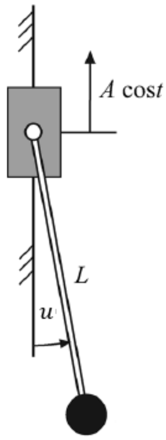

Figure 2. Mathematical pendulum with a vertically moving support (Kovacic et al., 2018)

When a linear control force is applied to stabilize such a system, the feedback mechanism inherently introduces a time delay. The equation of motion for this delayed system is given by:

$$u''(t) + (\delta + \varepsilon \cos t)u(t) = \zeta u(t - \tau) \quad (14)$$

where δ represents the square of the natural frequency, ε is the amplitude of parametric excitation, ζ is the delay coefficient, and τ is the time delay (taken as 2π).

4.1.1. Numerical implementation and comparison

The problem is solved over the domain $[0, 4\pi]$ using the PIA(1,1) algorithm with domain decomposition ($J = 2$). For parameters $\delta = 18$, $\varepsilon = 1$, and $\zeta = 0.5$, the semi-analytical solutions obtained by PIA are compared with the numerical results from the MATLAB dde23 solver.

As shown in Figure 3, the second and third iteration solutions of PIA exhibit excellent agreement with the numerical solution, confirming the high accuracy of the proposed method even for complex oscillatory behavior.

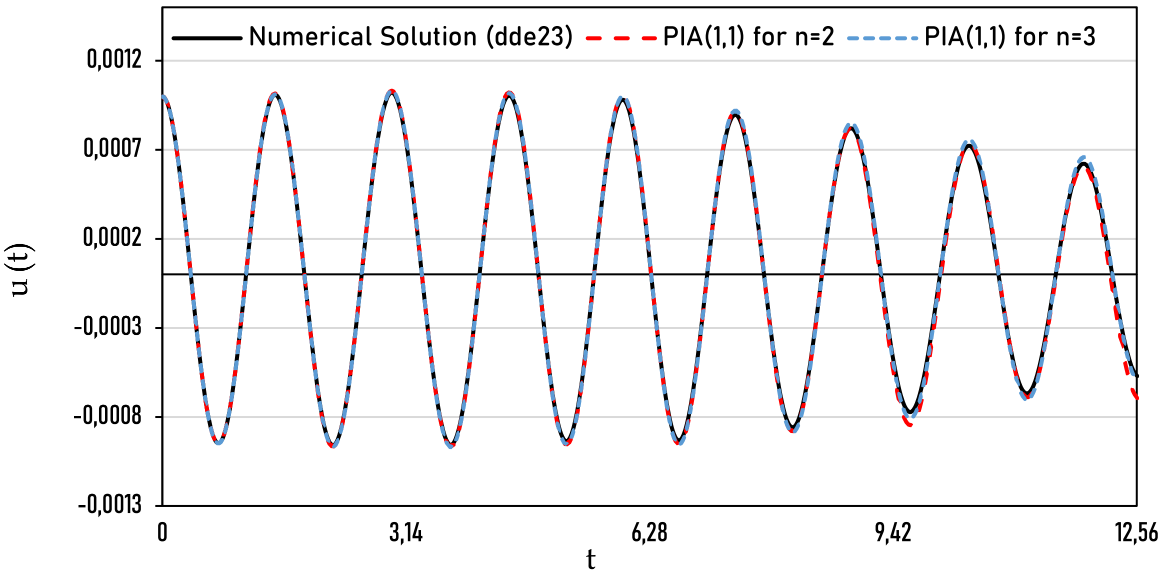

Figure 3. Comparison of the second and third iteration PIA(1,1) solutions with the MATLAB (dde23) numerical solution for the DME ($\delta = 18.0$, $\varepsilon = 1$, $\zeta = 0.5$, $\tau = 2\pi$)

4.1.2. Parametric analysis: Effect of delay and excitation

The influence of the delay coefficient (ζ) on the system's dynamics is analyzed in Figure 4. It is observed that increasing the delay coefficient (from $\zeta = 0$ to $\zeta = 1$) reduces the amplitude of oscillations. This phenomenon suggests that the delay term acts as a damping mechanism in the system. Mathematically, this can be approximated by the expansion $\zeta u(t-\tau) \approx \zeta u(t) - \zeta\tau u'(t)$, where the term $-\zeta\tau u'(t)$ introduces a velocity-dependent damping effect. It should be emphasized that this interpretation represents an approximate, first-order physical analogy intended to provide qualitative insight into the observed amplitude reduction, rather than an exact equivalence to classical viscous damping.

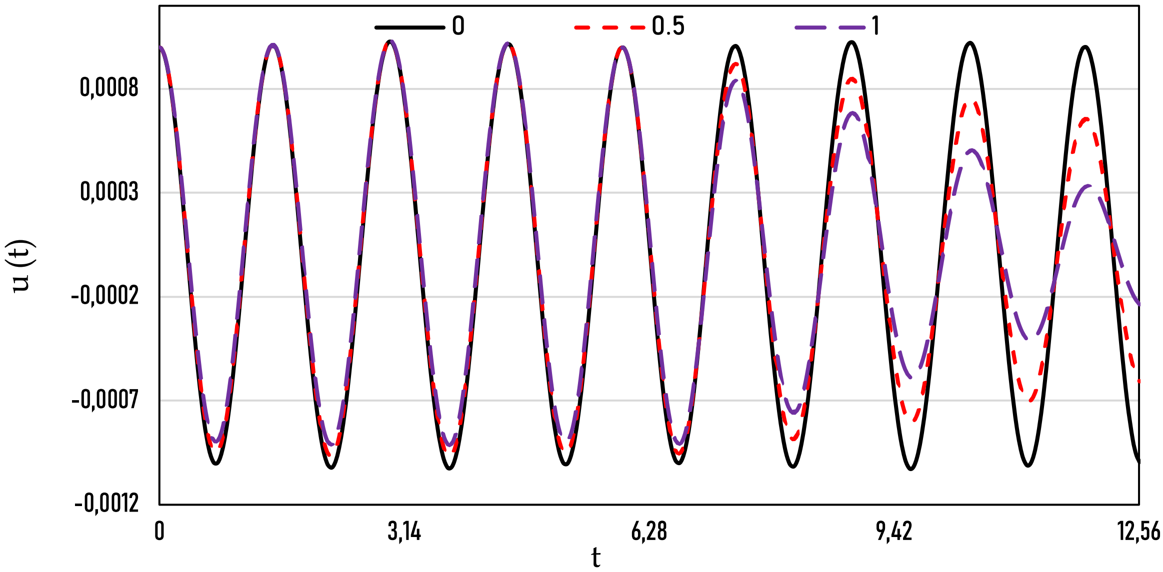

Figure 4. Fourth iteration PIA(1,1) solutions for different delay coefficients $\zeta = 0, 0.5, 1$ ($\delta = 18$, $\varepsilon = 1$, $\tau = 2\pi$)

The effect of the perturbation parameter ε, which represents the magnitude of the parametric excitation, is investigated in Figure 5 and Figure 6. As ε increases (e.g., from $\varepsilon = 1$ to $\varepsilon = 5$), the system becomes "stiffer," and the convergence rate of the algorithm decreases slightly, requiring higher iterations (e.g., $n = 6$) to match the numerical solution. Furthermore, as shown in Figure 6, increasing ε leads to a significant increase in the vibration amplitude.

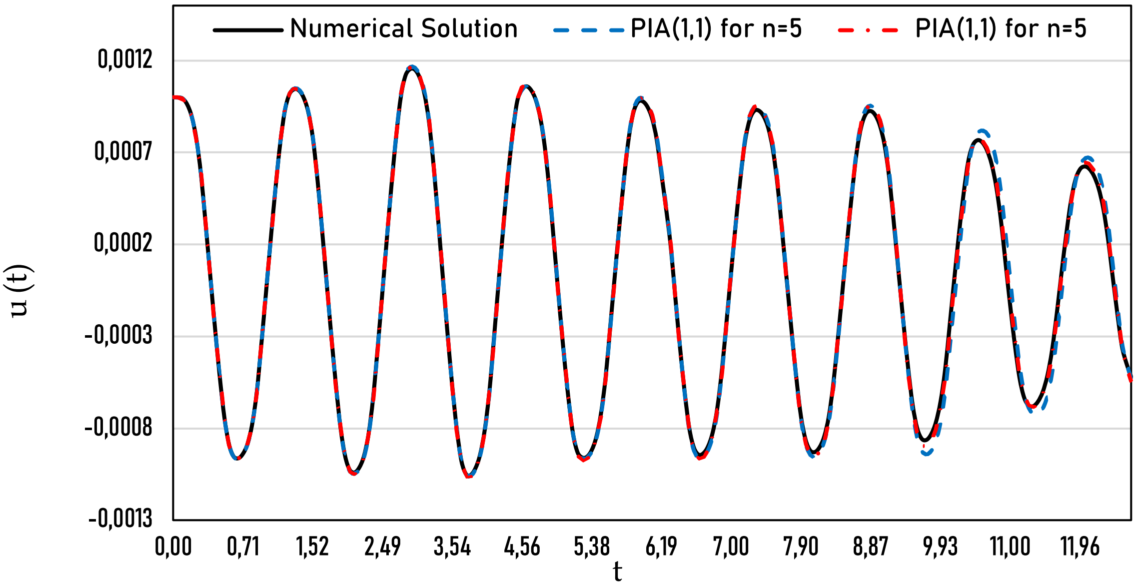

Figure 5. Comparison of PIA(1,1) solutions (5th and 6th iterations) with numerical results for high parametric excitation ε = 5

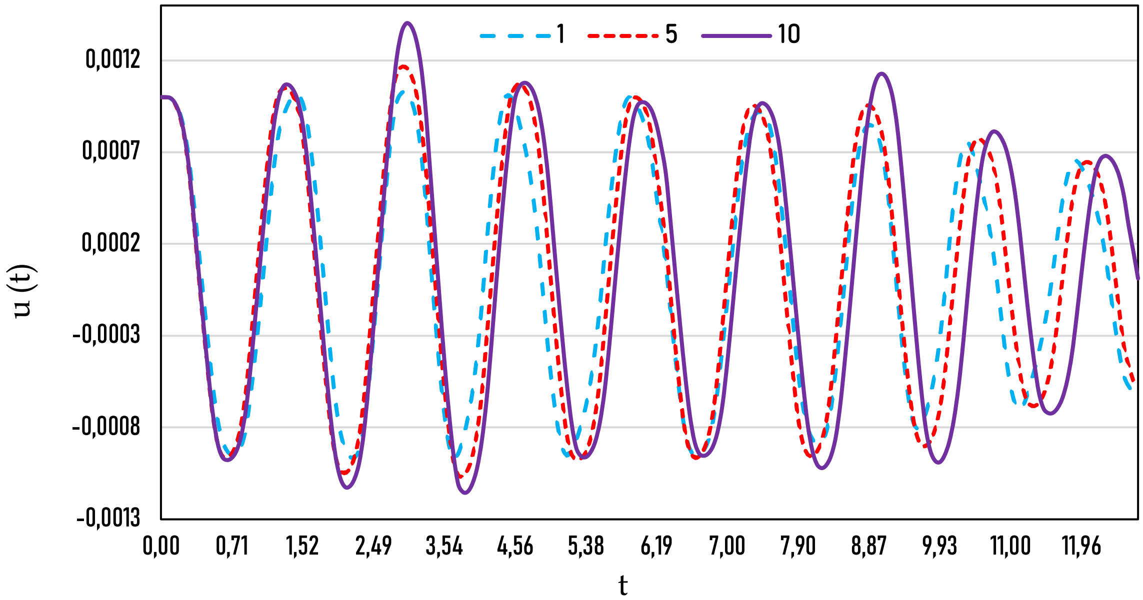

Figure 6. PIA(1,1) solutions for varying excitation amplitudes $\varepsilon = 1, 5, 10$ $(\delta = 18, \zeta = 0.5, \tau = 2\pi)$

Finally, the effect of the natural frequency parameter δ is presented in Figure 7. Variations in δ directly alter the oscillation period of the system, demonstrating the method's capability to capture frequency shifts accurately.

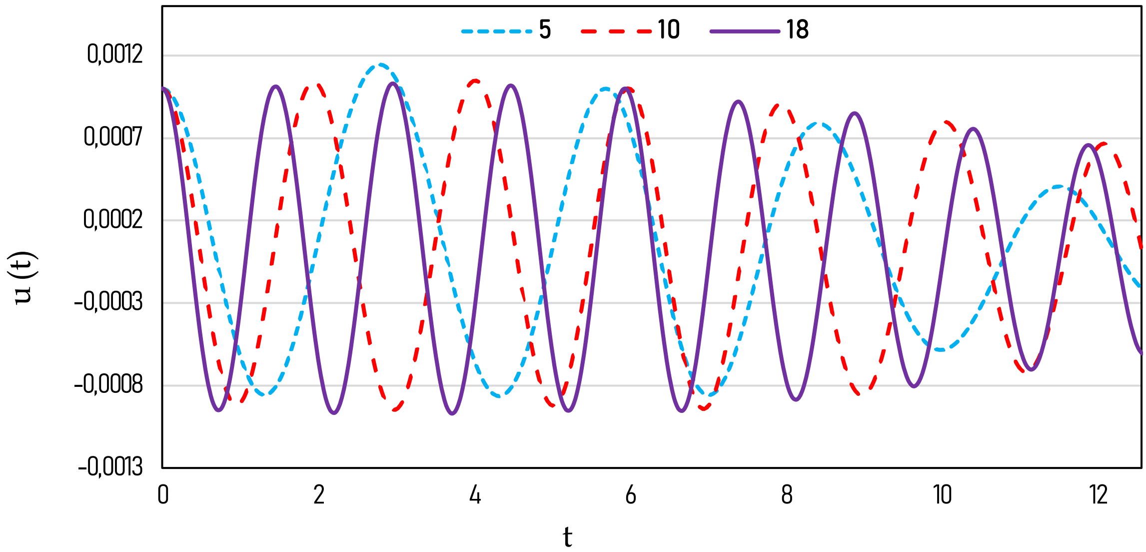

Figure 7. PIA(1,1) solutions for different δ values of 5, 10, and 18

4.2. Dynamic analysis of the DDME

To provide a more comprehensive analysis of engineering systems where both physical damping and time delays are present, we consider the DDME:

$$u''(t) + \kappa u'(t) + (\delta + \varepsilon \cos t)u(t) = \zeta u(t - \tau) \tag{15}$$

where κ is the physical damping coefficient.

The performance of the PIA(1,1) algorithm is compared against EMHPM (Olvera et al., 2015) and the standard numerical solver. Figure 8 presents the results for $\kappa = 0.2, \delta = 3.0, \varepsilon = 1$, and $\zeta = -1$. The comparison reveals that the PIA solution aligns more closely with the exact numerical solution than the EMHPM solution, highlighting the superior convergence properties of the proposed algorithm for damped delayed systems.

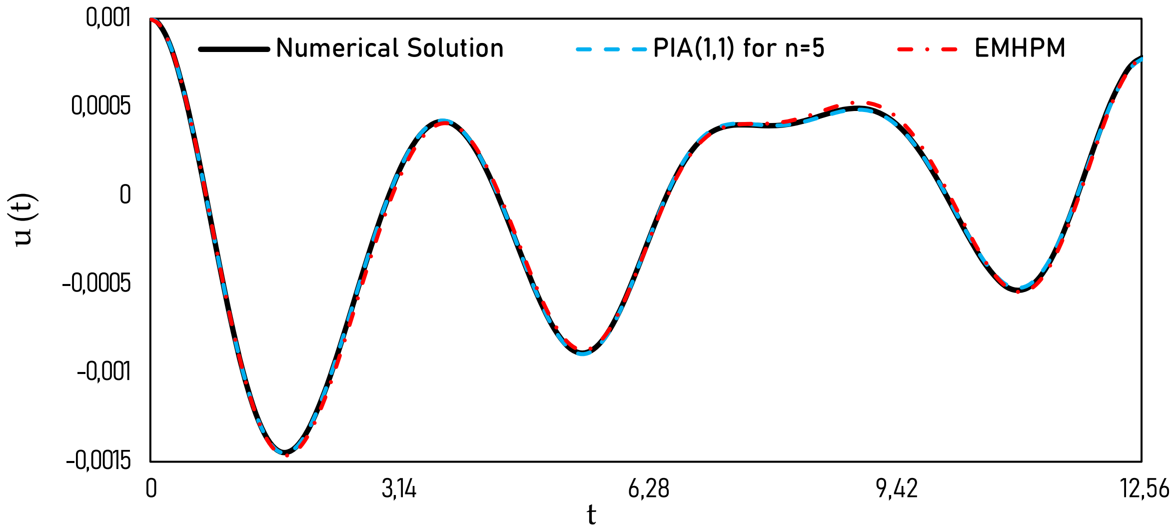

Figure 8. Comparison of PIA(1,1) (5th iteration), EMHPM (1st order), and numerical solutions for the DDME

The effect of the physical damping coefficient κ is illustrated in Figure 9. As expected, increasing κ rapidly decays the oscillation amplitude, validating the method's handling of the first-derivative term.

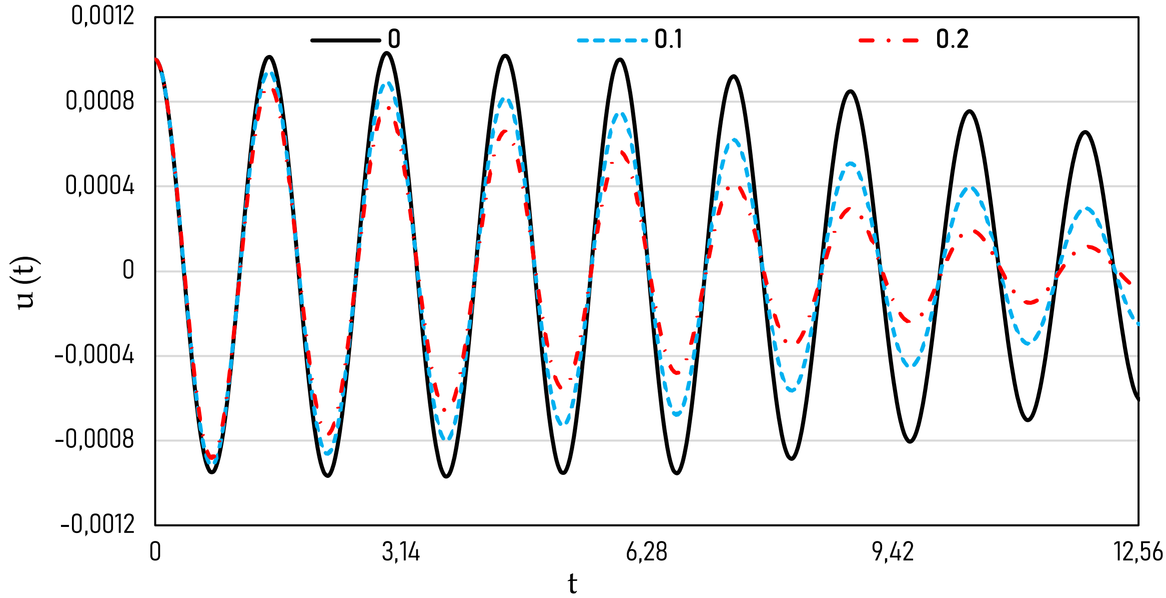

Figure 9. Fifth iteration PIA(1,1) solutions for different damping coefficients $\kappa = 0, 0.1, 0.2$

A critical analysis is performed in Figure 10, which examines the interplay between physical damping and the delay coefficient (ζ). While positive values of ζ (e.g., 0.5) reinforce the damping effect, a negative delay coefficient (e.g., $\zeta = -0.87$) counteracts the physical damping. This leads to a scenario where the system maintains a certain amplitude or approaches instability, despite the presence of physical damping. This capability to predict stability boundaries is crucial for the design of active control systems.

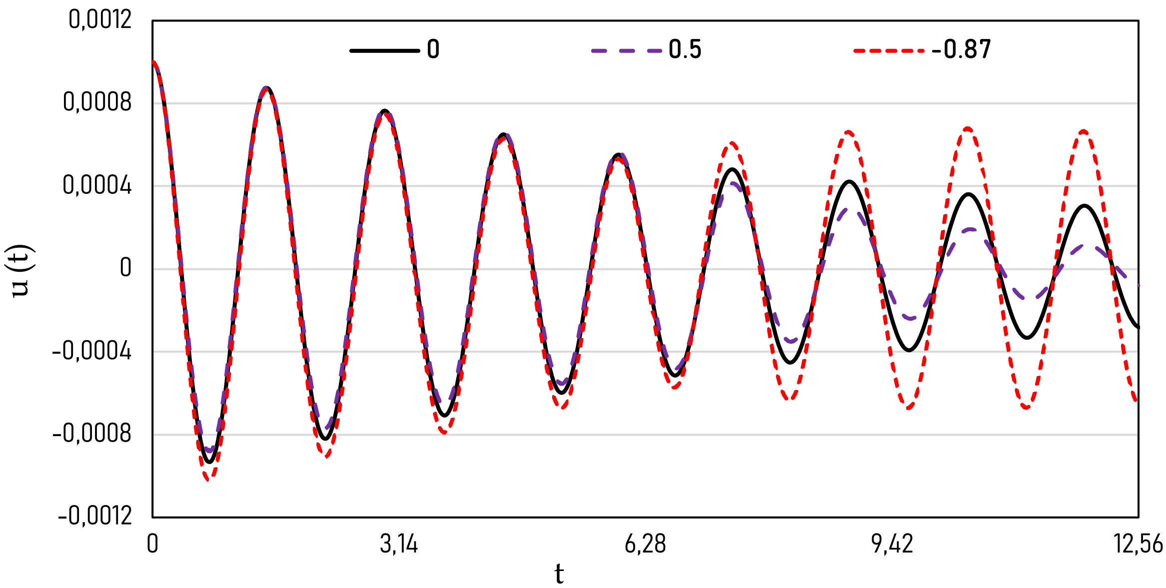

Figure 10. PIA(1,1) solutions for varying delay coefficients $\zeta = 0, 0.5, -0.87$ under physical damping $\kappa = 0.2$

5. Conclusion

In this study, the PIA(1,1) was successfully extended and applied to solve high-order delay differential equations. By integrating the method of steps with a robust continuity enforcement procedure based on matrix operations, the algorithm effectively manages the history functions and time delays inherent in such systems.

Numerical validation on a linear benchmark problem demonstrated that the proposed PIA(1,1) algorithm outperforms widely used techniques in the literature, such as the VMM and the DAM, in both solution accuracy and computational efficiency. The method has shown its suitability for computationally intensive engineering problems by achieving high-precision results with minimal CPU time.

Application of the method to the DME provided valuable physical insights into the stability of parametrically excited systems. Analysis of the results indicates that the delay term acts as an "artificial" damping mechanism; specifically, increasing the positive delay coefficient significantly reduces vibration amplitudes, thereby stabilizing the system. Conversely, investigations of the DDME revealed that negative delay coefficients act as an energy source counteracting physical damping, potentially driving the system towards instability. Furthermore, comparative analyses indicated that the PIA algorithm produces results closer to the exact numerical solution than the EMHPM, particularly for complex damped systems. In conclusion, this study confirms that the proposed PIA(1,1) algorithm is a powerful, reliable, and efficient mathematical tool for the dynamic analysis and design of linear and nonlinear engineering systems with time delays.

Despite its accuracy and efficiency for constant-delay problems, the proposed method is primarily suited to delay differential equations with fixed delays and moderate nonlinearity. For strongly nonlinear or high-dimensional systems, the size of the resulting algebraic systems and the computational cost may increase, potentially affecting efficiency. Moreover, extensions of the present framework to problems involving state-dependent delays are not considered in this study and remain an important direction for future research.

Authorship Contribution Statement

The author is solely responsible for the conceptualization, methodology, data collection, analysis, and manuscript preparation.

Conflict of Interest

The author declares no conflict of interest.

References

Alenazy, A. H. S., Ebaid, A., Algehyne, E. A., & Al-Jeaid, H. K. (2022). Advanced study on the delay differential equation $y'(t) = ay(t) + by(ct)$. *Mathematics*, *10*(22), 4302. https://doi.org/10.3390/math10224302

Amirali, I. (2023). Stability properties for the delay integro-differential equation. *Gazi University Journal of Science*, *36*(2), 862–868. https://doi.org/10.35378/gujs.988728

Aziz, I., & Amin, R. (2016). Numerical solution of a class of delay differential and delay partial differential equations via Haar wavelet. *Applied Mathematical Modelling*, *40*(23–24), 10286–10299. https://doi.org/10.1016/j.apm.2016.06.001

Blanco-Cocom, L., Estrella, A. G., & Avila-Vales, E. (2012). Solving delay differential systems with history functions by the Adomian decomposition method. *Applied Mathematics and Computation*, *218*(10), 5994–6011. https://doi.org/10.1016/j.amc.2011.11.082

Bahşı, M. M., & Bahşı, A. K. (2025). Numerical solution of the pantograph type differential equation using Jacobi neural network. In *Proceedings of the 5th International Artificial Intelligence and Data Science Congress (ICADA 2025)* (pp. 514–522).

Bahşi, M. M., & Çevik, M. (2015). Numerical solution of pantograph-type delay differential equations using perturbation-iteration algorithms. *Journal of Applied Mathematics, 2015*, Article 139821. https://doi.org/10.1155/2015/139821

Bahşı, M. M., Çevik, M., & Sezer, M. (2015). Orthoexponential polynomial solutions of delay pantograph differential equations with residual error estimation. *Applied Mathematics and Computation, 271*, 11–21. https://doi.org/10.1016/j.amc.2015.08.101

Barde, A., & Maan, N. (2019). Natural homotopy analysis method for nonlinear delay differential equations. *Science Proceedings Series, 1*(2), 86–90.

Cayan, S., Özhan, B. B., & Sezer, M. (2022). A Taylor-splitting collocation approach and applications to linear and nonlinear engineering models. *Chaos, Solitons & Fractals, 164*, 112683. https://doi.org/10.1016/j.chaos.2022.112683

Çevik, M., Savaşaneril, N. B., & Sezer, M. (2025). A review of polynomial matrix collocation methods in engineering and scientific applications. *Archives of Computational Methods in Engineering, 32*(8), 3355–3373. https://doi.org/10.1007/s11831-025-10235-6

Dağ, H. G., & Biçer, K. E. (2026). Boole polynomial approach for solving nonlinear delay differential equations. In *Systems science and nonlinear intelligence dynamics: Nonlinear dynamical control, computer simulation and optimization systems* (pp. 217–225). World Scientific. https://doi.org/10.1142/9789819815432_0020

Ford, N. J. (2025). Mathematical modelling of problems with delay and after-effect. *Applied Numerical Mathematics, 208* (Part B), 338–347. https://doi.org/10.1016/j.apnum.2024.10.007

Khuri, S. A., & Sayfy, A. (2017). Numerical solution of functional differential equations: A Green's function-based iterative approach. *International Journal of Computer Mathematics, 95*(10), 1937–1949. https://doi.org/10.1080/00207160.2017.1344230

Kovacic, I., Rand, R., & Mohamed Sah, S. (2018). Mathieu's equation and its generalizations: Overview of stability charts and their features. *Applied Mechanics Reviews, 70*(2), Article 020802. https://doi.org/10.1115/1.4039144

Kumar, A., & Methi, G. (2021). An efficient numerical algorithm for solution of nonlinear delay differential equations. *Journal of Physics: Conference Series, 1849*, Article 012014. https://doi.org/10.1088/1742-6596/1849/1/012014

Martín, J. A., & García, O. (2002). Variable multistep methods for higher-order delay differential equations. *Mathematical and Computer Modelling, 36*(7–8), 805–820. https://doi.org/10.1016/S0895-7177(02)00228-5

Ohira, K., & Ohira, T. (2025). Solving a delay differential equation through the Fourier transform. *Physics Letters A, 531*, 130138. https://doi.org/10.1016/j.physleta.2024.130138

Olvera, D., Elías-Zúñiga, A., López de Lacalle, L. N., & Rodríguez, C. A. (2015). Approximate solutions of delay differential equations with constant and variable coefficients by the enhanced multistage homotopy perturbation method. *Abstract and Applied Analysis, 2015*, Article 382475. https://doi.org/10.1155/2015/382475

Polyanin, A. D., Sorokin, V. G., & Zhurov, A. I. (2023). *Delay ordinary and partial differential equations* (1st ed.). Chapman and Hall/CRC. https://doi.org/10.1201/9781003042310

Rihan, F. A. (2021). *Delay differential equations and applications to biology*. Springer Nature Singapore. https://doi.org/10.1007/978-981-16-0626-7

Sun, P., Zhao, X., Yu, X., Huang, Q., Feng, Z., & Zhou, J. (2023). Incremental harmonic balance method for multi-harmonic solution of high-dimensional delay differential equations: Application to crossflow-induced nonlinear vibration of steam generator tubes. *Applied Mathematical Modelling, 118*, 818–831. https://doi.org/10.1016/j.apm.2023.02.018

Seong, H. Y., Majid, Z. A., & Ismail, F. (2013). Solving second-order delay differential equations by direct Adams-Moulton method. *Mathematical Problems in Engineering, 2013*, Article 261240. https://doi.org/10.1155/2013/261240

Yang, Y., Yuan, J.-W., Tie, D., Wan, M., & Zhang, W.-H. (2023). An efficient and accurate chatter prediction method of milling processes with a transition matrix reduction scheme. *Mechanical Systems and Signal Processing, 182*, 109535. https://doi.org/10.1016/j.ymssp.2022.109535

Yüzbaşı, Ş., & Ismailov, N. (2018). A Taylor operation method for solutions of generalized pantograph type delay differential equations. *Turkish Journal of Mathematics, 42*(2), 395–406. https://doi.org/10.3906/mat-1506-71