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Some Compact Operators on the Hahn Space

Eberhard Malkowsky[1,*]

[1]University Union Nikola Tesla, Faculty of Management, Belgrade, Serbia
[*]*Corresponding author: Eberhard.Malkowsky@math.uni-giessen.de; ema@pmf.ni.ac.rs*

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Abstract

We establish the characterisations of the classes of bounded linear operators from the generalised Hahn sequence space h_d, where d is an unbounded monotone increasing sequence of positive real numbers, into the spaces $[c_0]$, $[c]$ and $[c_\infty]$ of sequences that are strongly convergent to zero, strongly convergent and strongly bounded. Furthermore, we prove estimates for the Hausdorff measure of noncompactness of bounded linear operators from h_d into $[c]$, and identities for the Hausdorff measure of noncompactness of bounded linear operators from h_d to $[c_0]$, and use these results to characterise the classes of compact operators from h_d to $[c]$ and $[c_0]$.

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1. Introduction and Notations

We use the standard notations ω for the set of all complex sequences $x = (x_k)_{k=1}^\infty$, and ℓ_∞, c, c_0 and ϕ for the sets of all bounded, convergent, null and finite sequences, that is, sequences terminating in zeros. We also write $e = (e_k)_{k=1}^\infty$ and $e^{(n)} = (e_k^{(n)})_{k=1}^\infty$ $(n \in \mathbb{N})$ for the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

We recall that a BK space X is a Banach sequence space with continuous coordinates $P_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $P_n(x) = x_n$ for all $x = (x_k)_{k=1}^\infty \in X$. A BK space $X \supset \phi$ is said to have AK if $x = \lim_{m\to\infty} x^{[m]}$ for all $x = (x_k)_{k=1}^\infty \in X$, where $x^{[m]} = \sum_{k=1}^m x_k e^{(k)}$ denotes the m–section of the sequence x. It is well known that the sets ℓ_∞, c, and c_0 are BK spaces with their natural norms $\|x\|_\infty = \sup_k |x_k|$, c_0 has AK, every sequence $x = (x_k)_{k=1}^\infty \in c$ has a unique representation $x = \xi e + \sum_{k=1}^\infty (x_k - \xi)e^{(k)}$, where $\xi = \lim_{k\to\infty} x_k$, and finally, ℓ_∞ is not separable and consequently has no Schauder basis. Let $X \subset \omega$. Then the set $X^\beta = \{a \in \omega : \sum_{k=1}^\infty a_k x_k$ converges for all $x \in X\}$ is the *β–dual of* X. Let $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of complex numbers, $A_n = (a_{nk})_{k=1}^\infty$ and $A^k = (a_{nk})_{n=1}^\infty$ be the sequences in the n^{th} row and the k^{th} column of A, and X and Y be subsets of ω. Then we write $A_n x = \sum_{k=1}^\infty a_{nk} x_k$ and $Ax = (A_n x)_{n=1}^\infty$ for $x = (x_k)_{k=1}^\infty$ provided all the series converge. The set $X_A = \{x \in \omega : Ax \in X\}$ is called the *matrix domain of* A *in* X, and (X, Y) denotes the class of all matrix transformations from X into Y, that is, $A \in (X, Y)$ if and only if $X \subset Y_A$.

The reader interested in the theory of sequence spaces and matrix transformations is referred to the monographs Boos (2000); de Malafosse et al. (2021); Kamthan and Gupta (1981); Malkowsky and Rakočević (2019); Ruckle (1981); Wilansky (1984); Zeller and Beekmann (1968).

If X and Y are Banach spaces, we use the standard notation $\mathcal{B}(X,Y)$ for the Banach space of all bounded linear operators $L : X \to Y$ with the operator norm $\|L\| = \sup\{|L(x)| : \|x\| = 1\}$; the space $X^* = \mathcal{B}(X,\mathbb{C})$ is called the continuous dual of X; its norm is $\|f\| = \sup\{|f(x)| : \|x\| = 1\}$ for all $f \in X^*$. Also $\mathcal{K}(X,Y)$ denotes the class of all compact operators in $\mathcal{B}(X,Y)$.

The following well–known result gives the relation between (X,Y) and $\mathcal{B}(X,Y)$.

Proposition 1.1 *Let X and Y be BK spaces.*

(a) If $A \in (X,Y)$, then $L_A \in \mathcal{B}(X,Y)$, where $L_A(x) = Ax$ for all $x \in X$, that is, matrix maps between BK spaces are continuous (Wilansky, 1984, Theorem 4.2.8).

(b) If X has AK, then every operator $L \in \mathcal{B}(X,Y)$ can be represented by a matrix $A \in (X,Y)$ such that

$$Ax = L(x) \text{ for all } x \in X \text{ (Jarrah and Malkowsky, 2003, Theorem 1.9).} \tag{1.1}$$

The operator $\Delta : \omega \to \omega$ of the so–called forward differences is defined by $\Delta x_k = x_k - x_{k+1}$ $(k = 1,2,\dots)$. The set $h = \{x \in \omega : \sum_{k=1}^\infty k|\Delta x_k| < \infty\} \cap c_0$ was defined by Hahn in 1922 (see Hahn (1922)) in connection with the theory of singular integrals; Hahn showed that h is a BK space with $\|x\|' = \sum_{k=1}^\infty k|\Delta x_k| + \sup_k |x_k|$ for all $x = (x_k)_{k=1}^\infty \in h$. Rao (1990) showed that the Hahn space is a BK space with AK with the norm $\|x\| = \sum_{k=1}^\infty k|\Delta x_k|$ for all $x = (x_k)_{k=1}^\infty \in h$.

Goes (1972) introduced and studied the generalised Hahn space h_d for arbitrary complex sequences $d = (d_k)_{k=1}^\infty$ with $d_k \neq 0$ for all k by $h_d = \{x \in \omega : \sum_{k=1}^\infty |d_k| \cdot |\Delta x_k| < \infty\} \cap c_0$ with the norm

$$\|x\|_d = \sum_{k=1}^\infty |d_k| \cdot |\Delta x_k| \text{ for all } x = (x_k)_{k=1}^\infty \in h_d. \tag{1.2}$$

The following result is known.

Proposition 1.2 *Let d be a increasing unbounded sequence of positive reals.*

(a) Then h_d with the norm in (1.2) *is a BK space with AK* (Malkowsky et al., 2021, Proposition 2.1).

(b) We write

$$bs_d = \left\{a \in \omega : \sup_n \frac{1}{d_n}\left|\sum_{k=1}^n a_k\right|\right\} \text{ and } \|a\|_{bs_d} = \sup_n \frac{1}{d_n}\left|\sum_{k=1}^n a_k\right| \text{ for all } a \in bs_d.$$

Then $h_d^\beta = bs_d$ and h_d^β and h_d^* are norm isomorphic (Malkowsky et al., 2021, Proposition 2.3).

Recent research on the Hahn space and its generalisations can be found, for instance, in Das (2017); Kirişci (2013a); Raj and Kiliçman (2014); Rao and Srinivasalu (1996); Rao and Subramanian (2002) and the survey paper Kirişci (2013b).

Throughout, we use the convention that every term with a subscript ≤ 0 is equal to zero. The sets

$$[c_0] = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |kx_k - (k-1)x_{k-1}| = 0 \right\},$$
$$[c] = [c_0] \oplus e = \{x \in \omega : x - \xi e \in [c_0] \text{ for some } \xi \in \mathbb{C}\} \text{ and}$$
$$[c_\infty] = \left\{ x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^{n} |kx_k - (k-1)x_{k-1}| < \infty \right\}$$

of sequences that are strongly convergent to zero, strongly convergent and strongly bounded were first introduced and studied in the papers Borwein (1960), and Kuttner and Thorpe (1979). Generalizations of these spaces were considered by Mòricz (1989), the research papers Djolović and Malkowsky (2012, 2013); Jarrah and Malkowsky (2002); Malkowsky (1995, 2000, 2013); Malkowsky and Nergiz (2015); Malkowsky and Rakočević (1998, 2000), and the survey paper Malkowsky (2017).

The following result is well–known.

Proposition 1.3 (Malkowsky, 1995, Theorem 2) *The sets $[c_0]$, $[c]$ and $[c_\infty]$ are BK spaces with*

$$\|x\|_{[c_\infty]} = \sup_n \frac{1}{n} \sum_{k=1}^{n} |kx_k - (k-1)x_{k-1}|;$$

$[c_0]$ is a closed subspace of $[c]$, and $[c]$ is a closed subspace of $[c_\infty]$; $[c_0]$ has AK and every sequence $x = (x_k)_{k=1}^\infty \in [c]$ has a representation

$$x = \xi e + \sum_{k=1}^{\infty} (x_k - \xi) e^{(k)}, \tag{1.3}$$

where ξ is the unique complex number such that $x - \xi e \in [c_0]$, the so–called $[c]$–limit of *x.*

In this paper, we characterise the classes $\mathcal{B}(h_d, [c_0])$, $\mathcal{B}(h_d, [c])$ and $\mathcal{B}(h_d, [c_\infty])$, when d is a monotone increasing unbounded sequence of positive real numbers. Furthermore, we establish estimates for the Hausdorff measure of noncompactness of operators in the class $\mathcal{B}(h_d, [c])$, and identities for the Hausdorff measure of noncompactness of operators in the class $\mathcal{B}(h_d, [c_0])$. Finally, we characterise the classes $\mathcal{K}(h_d, [c])$ and $\mathcal{K}(h_d, [c_0])$.

2. The Classes $\mathcal{B}(h_d, Y)$ for $Y \in \{[c_\infty], [c], [c_0]\}$

Throughout let d be an unbounded increasing sequence of positive real numbers.

We are going to characterise the classes $\mathcal{B}(h_d, Y)$ and compute the operator norm of $L \in \mathcal{B}(h_d, Y)$ for $Y \in \{[c_\infty], [c], [c_0]\}$. Since h_d is a BK space with AK by Proposition 1.2 (a), and each space Y is a BK space by Proposition 1.3, each operator $L \in \mathcal{B}(h_d, Y)$ can be represented by a matrix $A \in (h_d, Y)$ as in (1.1) by Proposition 1.1 (b). We will use this fact and notation throughout the paper.

We need the following definition and results which we state here for the reader's convenience.

Definition 2.1 (Wilansky, 1984, Definition 7.4.2) Let X be a BK space. A subset E of the set ϕ called a *determining set for X* if $D(X) = \bar{B}_X \cap \phi$ is the absolutely convex hull of E.

Proposition 2.2 (Wilansky, 1984, Theorem 8.3.4) *Let X be a BK space with AK, E be a determining set for X, and Y be an FK space. Then $A \in (X, Y)$ if and only if:*

(i) The columns of A belong to Y, that is, $A^k = (a_{nk})_{n=1}^{\infty} \in Y$ for all k,

and

(ii) $L(E)$ is a bounded subset of Y,where $L(x) = Ax$ for all $x \in X$.

Proposition 2.3 (Malkowsky et al., 2021, Proposition 3.2) *The set*

$$E = \left\{ \frac{1}{d_m} \cdot e^{[m]} : m \in \mathbb{N} \right\} \tag{2.1}$$

is a determinig set for h_d.

Theorem 2.4 *We have*
(a) $L \in \mathcal{B}(h_d, [c_\infty])$ if and only if

$$\|A\|_{(h_d,[c_\infty])} = \sup_{l,m} \frac{1}{l d_m} \sum_{n=1}^{l} \left| n \sum_{k=1}^{m} a_{nk} - (n-1) \sum_{k=1}^{m} a_{n-1,k} \right| < \infty; \tag{2.2}$$

(b) $L \in \mathcal{B}(h_d, [c])$ if and only if (2.2) holds and

$$\left\{ \begin{array}{c} \text{for each } k \in \mathbb{N}, \text{ there exists } \alpha_k \in \mathbb{C} \text{ such that} \\ \lim_{l \to \infty} \frac{1}{l} \sum_{n=1}^{l} |n a_{nk} - (n-1) a_{n-1,k} - \alpha_k| = 0; \end{array} \right\} \tag{2.3}$$

(c) $L \in \mathcal{B}(h_d, [c_0])$ if and only if (2.2) holds and

$$\lim_{l \to \infty} \frac{1}{l} \sum_{n=1}^{l} |n a_{nk} - (n-1) a_{n-1,k}| = 0 \text{ for each } k. \tag{2.4}$$

(d) If $L \in \mathcal{B}(h_d, Y)$ for $Y \in \{[c_0], [c], [c_\infty]\}$, then

$$\|L\| = \|A\|_{(h_d,[c_\infty])}. \tag{2.5}$$

Proof. (a) Let $L \in \mathcal{B}(h_d, [c_\infty])$.
Since the set E in (2.1) is a determining set for h_d by Proposition 2.3, we apply Proposition 2.2, and show that the matrix A that represents L satisfies the conditions in (i) and (ii) of Proposition 2.2.
We write $C = (c_{nm})_{n,m=1}^{\infty}$ for the matrix with

$$c_{nm} = n \sum_{k=1}^{m} a_{nk} \text{ for } n, m = 1, 2, \ldots$$

and

$$\Delta_n^- c_{nm} = c_{nm} - c_{n-1,m} \text{ for } n, m = 1, 2, \ldots.$$

Let $m \in \mathbb{N}$ be given and $y^{(m)} = (1/d_m) e^{[m]} \in E$. Then we have

$$A_n y^{(m)} = \sum_{k=1}^{\infty} a_{nk} y_k^{(m)} = \frac{1}{d_m} \sum_{k=1}^{m} a_{nk} = \frac{1}{n d_m} c_{nm} \text{ for all } n,$$

hence

$$\begin{aligned}\left\|Ay^{(m)}\right\|_{[c_\infty]} &= \sup_l \frac{1}{l}\sum_{n=1}^{l}\left|nA_n y^{(m)} - (n-1)A_{n-1}y^{(m)}\right| \\ &= \sup_l \frac{1}{ld_m}\sum_{n=1}^{l}\left|\Delta_n^- c_{nm}\right| \le \|A\|_{(h_d,[c_\infty])} < \infty.\end{aligned}$$

So (2.2) yields the condition in (ii) of Proposition 2.2.
It remains to show that the condition in (i) of Proposition 2.2 is redundant.
We have

$$a_{nm} = d_m A_n y^{(m)} - d_{m-1}A_n y^{(m-1)} = \frac{1}{n}(c_{nm} - c_{n,m-1}) \text{ for all } n \text{ and } m,$$

hence

$$|na_{nm} - (n-1)a_{n-1,m}| = |c_{nm} - c_{n,m-1} - (c_{n-1,m} - c_{n-1,m-1}| = \left|\Delta_n^- c_{nm}\right| + \left|\Delta_n^- c_{n,m-1}\right|$$

and

$$\begin{aligned}\|A^m\|_{[c_\infty]} &= \sup_l \frac{1}{l}\sum_{n=1}^{l}|na_{nm} - (n-1)a_{n,m-1}| \\ &\le d_m \sup_l \frac{1}{ld_m}\sum_{n=1}^{l}\left|\Delta_n^- c_{nm}\right| + d_{m-1}\sup_l \frac{1}{ld_{m-1}}\sum_{n=1}^{l}\left|\Delta_n^- c_{n,m-1}\right| \\ &\le 2d_m \cdot \|A\|_{(h_d,[c_\infty])} < \infty \text{ for all } m.\end{aligned}$$

This completes the proof of Part (a).
(b) and (c) Since h_d is a BK space with AK and $[c_0]$ and $[c]$ are closed subspaces of the BK space $[c_\infty]$ by Proposition 1.3, Parts (b) and (c) follow by (Wilansky, 1984, Theorem 8.3.6).
(d) Finally we assume that $L \in \mathcal{B}(h_d, Y)$, where $Y \in \{[c_0],[c],[c_\infty]\}$. Then $A_n \in h_d^\beta$ for all n and $h_d^\beta = bs_d$ by Proposition 1.2 (b). We obtain for $A_n x = L_n(x)$ $(x \in h_d)$

$$|A_n x| \le \sum_{k=1}^{\infty} d_k|\Delta x_k|\frac{1}{d_k}\left|\sum_{j=1}^{k} a_{nj}\right| \le \|A_n\|_{bs_d} \cdot \|x\|_{h_d} \text{ for all } n \in \mathbb{N} \text{ and all } x \in h_d. \tag{2.6}$$

To prove (2.6), let $m \in \mathbb{N}$ be given. Then Abel's summation by parts yields

$$\begin{aligned}L_n(x^{[m]}) = A_n x^{[m]} &= \sum_{k=1}^{m} a_{nk}x_k = \sum_{k=1}^{m-1}\Delta x_k \sum_{j=1}^{k} a_{nj} + x_m \sum_{j=1}^{m} a_{nj} \\ &= \sum_{k=1}^{m-1} d_k \Delta x_k \frac{1}{d_k}\sum_{j=1}^{k} a_{nj} + d_m x_m \frac{1}{d_m}\sum_{j=1}^{m} a_{nj}.\end{aligned}$$

Since h_d has AK and $x \in h_d$, it follows that

$$\begin{aligned}0 \le |d_m x_m| &= \sum_{k=m}^{\infty} d_k\left|\Delta x_k^{[m]}\right| \le \sum_{k=1}^{\infty} d_k\left|\Delta(x_k^{[m]} - x_k)\right| + \sum_{k=m}^{\infty} d_k|\Delta x_k| \\ &= \|x^{[m]} - x\|_{h_d} + \sum_{k=m}^{\infty} d_k|\Delta x_k| \to 0 \ (m \to \infty).\end{aligned}$$

Thus the continuity of L_n yields

$$|A_n x| = |L_n(x)| = \lim_{m\to\infty}\left|L_n(x^{[m]})\right| \le \sum_{k=1}^{\infty} d_k|\Delta x_k| \frac{1}{d_k}\left|\sum_{j=1}^{k} a_{nj}\right|$$
$$\le \sum_{k=1}^{\infty} d_k|\Delta x_k| \|A_n\|_{bs_d} = \|A_n\|_{bs_d}\cdot\|x\|_{h_d},$$

that is, (2.6).

Now we write $B = (b_{nk})_{n,k=1}^{\infty}$ for the matrix with the rows $B_n = nA_n - (n-1)A_{n-1}$ for $n = 1, 2, \dots$. Then we obtain from (2.6) for $l = 1, 2, \dots$

$$\begin{aligned}
\frac{1}{l}\sum_{n=1}^{l}|nA_n x - (n-1)A_{n-1}x| &= \frac{1}{l}\sum_{n=1}^{l}|B_n x| \le \frac{1}{l}\sum_{n=1}^{l}\sum_{m=1}^{\infty} d_m|\Delta x_m|\frac{1}{d_m}\left|\sum_{j=1}^{m} b_{nj}\right| \\
&= \frac{1}{l}\sum_{m=1}^{\infty} d_m|\Delta x_m|\left(\frac{1}{d_m}\sum_{n=1}^{l}\left|\sum_{j=1}^{m} b_{nj}\right|\right) \\
&\le \frac{1}{l}\left(\sup_m \frac{1}{d_m}\sum_{n=1}^{l}\left|\sum_{j=1}^{m} b_{nj}\right|\right)\cdot\|x\|_{h_d} \\
&\le \left(\sup_{l,m}\frac{1}{l d_m}\sum_{n=1}^{l}\left|\sum_{j=1}^{m}(n a_{nj} - (n-1)a_{n-1,j})\right|\right)\|x\|_{h_d} \\
&\le \|A\|_{(h_d,[c_\infty])}\|x\|_{h_d}.
\end{aligned}$$

This implies

$$\|L\| = \sup_{\|x\|_{h_d}=1}\|L(x)\|_{[c_\infty]} = \sup_{\|x\|_{h_d}=1}\frac{1}{l}\sum_{n=1}^{l}|nA_n x - (n-1)A_{n-1}x| \le \|A\|_{(h_d,[c_\infty])}. \tag{2.7}$$

To prove the converse inequality, let $m \in \mathbb{N}$ be given. We put $x^{(m)} = (1/d_m)e^{[m]}$. Then

$$\|x^{(m)}\|_{h_d} = \frac{1}{d_m}\sum_{k=1}^{m} d_k\left|\Delta x_k^{(m)}\right| = \frac{d_m}{d_m} = 1$$

and

$$\begin{aligned}
\|L(x^{(m)})\|_{[c_\infty]} &= \sup_l \frac{1}{l}\sum_{n=1}^{l}\left|nA_n x^{(m)} - (n-1)A_{n-1}x^{(m)}\right| \\
&= \sup_l \frac{1}{l d_m}\sum_{n=1}^{l}|c_{nm} - c_{n-1,m}| \le \|L\|.
\end{aligned}$$

Since $m \in \mathbb{N}$ was arbitrary, we have $\|A\|_{(h_d,[c_\infty])} \le \|L\|$.
Finally, this and (2.7) together imply (2.5). □

Now we establish a formula for the $[c]$–limits of $L(x)$ and $x \in h_d$, when $L \in \mathcal{B}(h_d, w)$.

Theorem 2.5 *Let $L \in \mathcal{B}(h_d, [c])$ and α_k for $k \in \mathbb{N}$ be the complex numbers in* (2.3). *Then the $[c]$–limit $\eta(x)$ of $L(x)$ for each sequence $x \in h_d$ is given by*

$$\eta(x) = \sum_{k=1}^{\infty} \alpha_k x_k. \tag{2.8}$$

Proof. Let $L \in \mathcal{B}(h_d, [c])$. We write $B = (b_{nk})_{n,k=1}^{\infty}$ for the matrix with the rows $B_n = nA_n - (n-1)A_{n-1}$ for all n. First we show

$$(\alpha_k)_{k=1}^{\infty} \in bs_d. \tag{2.9}$$

We have for all $l, m \in \mathbb{N}$

$$\begin{aligned}\frac{1}{d_m}\left|\sum_{k=1}^{m} \alpha_k\right| &= \frac{1}{d_m} \cdot \frac{1}{l} \sum_{n=1}^{l} \left|\sum_{k=1}^{m} \alpha_k\right| \\ &\le \frac{1}{d_m} \cdot \frac{1}{l} \sum_{n=1}^{l} \left|\sum_{k=1}^{m} b_{nk} - \alpha_k\right| + \frac{1}{d_m} \cdot \frac{1}{l} \sum_{n=1}^{l} \left|\sum_{k=1}^{m} b_{nk}\right| \\ &\le \frac{1}{d_m} \sum_{k=1}^{m} \frac{1}{l} \sum_{n=1}^{l} |b_{nk} - \alpha_k| + \|A\|_{(h_d, [c_\infty])}. \end{aligned} \tag{2.10}$$

Since, for each fixed m, the first term in the last inequality above tends to 0 as l tends to infinity by (2.3), it follows that

$$\sup_m \frac{1}{d_m}\left|\sum_{k=1}^{m} \alpha_k\right| \le \|A\|_{(h_d, [c_\infty])} < \infty, \tag{2.11}$$

and so (2.9) is satisfied.

By Proposition 1.2 (b) and (2.9), we have $(\alpha_k)_{k=1}^{\infty} \in h_d^{\beta}$. Also $A \in (h_d, [c])$ implies $A_n \in h_d^{\beta}$ for each n, and consequently $B_n - (\alpha_k)_{k=1}^{\infty} \in h_d^{\beta}$ for each n. Now we obtain for all m and l by (2.11)

$$\frac{1}{ld_m} \sum_{n=1}^{l} \left|\sum_{k=1}^{m} (b_{nk} - \alpha_k)\right| \le \frac{1}{ld_m} \sum_{n=1}^{l} \left|\sum_{k=1}^{m} b_{nk}\right| + \frac{1}{ld_m} \sum_{n=1}^{l} \left|\sum_{k=1}^{m} \alpha_k\right| \le 2\|A\|_{(h_d, [c_\infty])} < \infty,$$

and so

$$\sup_{m,l} \frac{1}{ld_m} \sum_{n=1}^{l} \left|\sum_{k=1}^{m} na_{nk} - (n-1)a_{nk} - \alpha_k\right| < \infty,$$

that is, $(a_{nk} - \alpha_k)_{n,k=1}^{\infty} \in (h_d, [c_\infty])$ by Theorem 2.4 (a). Finally, this and (2.3) imply $(a_{nk} - \alpha_k)_{n,k=1} \in (h_d, [c_0])$, so the $[c]$–limit of $L(x)$ for $x \in h_d$ is given by (2.8). □

3. The Hausdorff Measure of Noncompactness of Operators

In this section, we establish an identity for the Hausdorff measure on noncompactness of operators in $\mathcal{B}(h_d, [c_0])$ and an estimate for the Hausdorff measure of noncompactness of operators in $\mathcal{B}(h_d, [c])$. We also characterise the classes $\mathcal{K}(h_d, [c_0])$ and $\mathcal{K}(h_d, [c])$.
We list the necessary, known concepts and results concerning the Hausdorff measure of noncompactness. First we recall the definition of the Hausdorff measure of noncompactness of bounded

sets in complete metric spaces (Toledano et al., 1997, Definition II.2.1), and the Hausdorff measure of noncompactness of operators between Banach spaces (Malkowsky and Rakočević, 2019, Definition 7.11.1). The interested reader is also referred to the research articles Mursaleen and Noman (2010, 2011).

Let X be a complete metric space and $\mathcal{M}_X$ be the class of bounded subsets of X. Then the function $\chi : \mathcal{M}_X \to [0, \infty)$ with $\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{–net in } X\}$ is called the *Hausdorff measure of noncompactness on X*.

Let χ_1 and χ_2 be Hausdorff measures of noncompactness on the Banach spaces X and Y, repectively. Then an operator $L : X \to Y$ is said to be (χ_1, χ_2)–bounded, if $L(Q) \in \mathcal{M}_Y$ for all $Q \in \mathcal{M}_X$ and there exists a non–negative real number c such that

$$\chi_2(L(Q)) \le c \cdot \chi_1(Q) \text{ for all } Q \in \mathcal{M}_X. \tag{3.1}$$

If an operator L is (χ_1, χ_2)–bounded, the the number

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{c \ge 0 : (3.1) \text{ is satisfied}\}$$

is called the (χ_1, χ_2)*–measure of noncompactness of the operator L*. If $\chi_1 = \chi_2$, we write $\|L\|_\chi = \|L\|_{(\chi,\chi)}$, for short, and refer to $\|L\|_\chi$ as the *Hausdorff measure of noncompactness of the operator L*.

We need the following known results.

Theorem 3.1 (Goldenštein, Gohberg, Markus) (Malkowsky and Rakočević, 2000, Theorem 2.23) *Let X be a Banach space with a Schauder basis (b_n), $\mathcal{R}_n : X \to X$ for each n be defined by*

$$\mathcal{R}_n(x) = \sum_{k=n+1}^{\infty} \lambda_k b_k \text{ for all } x = \sum_{k=1}^{\infty} \lambda_k b_k \in X,$$

and $\mu : \mathcal{M}_X \to \mathcal{M}_X$ be the function with

$$\mu(Q) = \limsup_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right).$$

Then

$$\frac{1}{a} \cdot \mu(Q) \le \chi(Q) \le \inf_n \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right) \le \mu(Q) \text{ for all } x \in \mathcal{M}_X, \tag{3.2}$$

where $a = \limsup_{n\to\infty} \|\mathcal{R}_n\|$ is the basis constant of the Schauder basis.

Proposition 3.2 *Let X and Y be Banach spaces and $L \in \mathcal{B}(X, Y)$ and S_X denote the unit sphere in X. Then we have*

$$\|L\|_\chi = \chi(L(S_X)) \text{ (Malkowsky and Rakočević, 2019, Theorem 7.11.4)} \tag{3.3}$$

and $L \in \mathcal{K}(X, Y)$ if and only if

$$\|L\|_\chi = 0 \text{ (Malkowsky and Rakočević, 2019, Theorem 7.11.5)}. \tag{3.4}$$

We obtain the follwoing results for the Hausdorff measure of noncompactness of bounded sets in $[c_0]$ and $[c]$.

Corollary 3.3 *We have*

$$\chi(Q) = \mu(Q) \text{ for all } Q \in \mathcal{M}_{[c_0]} \tag{3.5}$$

and

$$\frac{1}{2} \cdot \mu(Q) \leq \chi(Q) \leq \mu(Q) \text{ for all } Q \in \mathcal{M}_{[c]}. \tag{3.6}$$

Proof. We show $a = 1$ for $\mathcal{R}_n : [c_0] \to [c_0]$ and $a \leq 2$ for $\mathcal{R}_n : [c] \to [c]$. Then (3.5) and (3.6) follow from (3.2).

We have $x = \sum_{k=1}^{\infty} x_k e^{(k)}$ for all $x = (x_k)_{k=1}^{\infty} \in [c_0]$ by Proposition 1.3, hence for each $m \in \mathbb{N}$

$$\begin{aligned}
\|\mathcal{R}_m(x)\|_{[c_\infty]} &= \sup_{n \geq m+1} \frac{1}{n} \sum_{k=m+1}^{n} |kx_k - (k-1)x_{k-1}| \\
&= \sup_{n \geq m+1} \frac{1}{n} \left((m+1)|x_{m+1}| + \sum_{k=m+2}^{n} |kx_k - (k-1)x_{k-1}| \right) \\
&= \sup_{n \geq m+1} \frac{1}{n} \left(\left| \sum_{k=1}^{m+1} (kx_k - (k-1)x_{k-1}) \right| + \sum_{k=m+2}^{n} |kx_k - (k-1)x_{k-1}| \right) \\
&\leq \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} |kx_k - (k-1)x_{k-1}| = \|x\|_{[c_\infty]},
\end{aligned}$$

hence

$$\|\mathcal{R}_m(x)\|_{[c_\infty]} \leq \|x\|_{[c_\infty]}. \tag{3.7}$$

This implies $\|\mathcal{R}_m\| \leq 1$ for all m. Since $\mathcal{R}_m$ is a projector, we also have $\|\mathcal{R}_m\| \geq 1$ for all m. Thus we have shown $a = 1$.

By (1.3), every sequence $x = (x_k)_{k=1}^{\infty} \in [c]$ has a unique representation

$$x = \xi e + \sum_{k=1}^{\infty} (x_k - \xi) e^{(k)},$$

where ξ is the $[c]$–limit of the sequence x. Now we have

$$\begin{aligned}
\|\mathcal{R}_m(x)\|_{[c_\infty]} &= \sup_{n \geq m} \frac{1}{n} \left(|(m+1)(x_{m+1} - \xi)| + \sum_{k=m+2}^{n} |k(x_k - \xi) - (k-1)(x_k - \xi)| \right) \\
&\leq \sup_{n \geq m} \frac{1}{n} \Bigg(|((m+1) - (n - (m+1)))\xi| + |(m+1)x_{m+1}| \\
&\qquad + \sum_{k=m+2}^{n} |kx_k - (k-1)x_{k-1}| \Bigg) \\
&\leq |\xi| + \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} |kx_k - (k-1)x_{k-1}|
\end{aligned}$$

and (3.7) yields

$$\|\mathcal{R}_m\|_{[c_\infty]} \leq |\xi| + \|x\|_{[c_\infty]}. \tag{3.8}$$

We also obtain for all n

$$|\xi| = \frac{1}{n} \sum_{k=1}^{n} |\xi| \leq \frac{1}{n} \sum_{k=1}^{n} |kx_k - (k-1)x_{k-1} - \xi| + \frac{1}{n} \sum_{k=1}^{n} |kx_k - (k-1)x_{k-1}|$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}|kx_k-(k-1)x_{k-1}-\xi|+\|x\|_{[c_\infty]}.$$

Since ξ is the $[c]$–limit of the sequence x, the first term in the last inequality tends to 0 as n tends to ∞, and so $|\xi| \leq \|x\|_{[c_\infty]}$. Now (3.8) yields $\|\mathcal{R}_m(x)\|_{[c_\infty]} \leq 2\|x\|_{[c_\infty]}$ for all m, hence $a \leq 2$. □

Now we prove an estimate for $\|L\|_\chi$, if $L \in \mathcal{B}(h_d,[c])$, and an identity for $\|L\|_\chi$, if $L \in \mathcal{B}(h_d,[c_0])$.

Theorem 3.4 *(a) Let $L \in \mathcal{B}(h_d,[c])$. Then we have*

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{ld_m}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}(na_{nk}-(n-1)a_{n-1,k}-\alpha_k)\right|\right)\leq\|L\|_\chi$$
$$\leq\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{ld_m}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}(na_{nk}-(n-1)a_{n-1,k}-\alpha_k)\right|\right),\quad(3.9)$$

where the complex numbers α_k are defined in (2.3).
(b) Let $L \in \mathcal{B}(h_d,[c_0])$. Then we have

$$\|L\|_\chi=\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{ld_m}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}(na_{nk}-(n-1)a_{n-1,k}\right|\right).\quad(3.10)$$

Proof. Let $A=(a_{nk})_{n,k=1}^\infty$ be any infinite matrix and $r\in\mathbb{N}$. We write $A^{<r>}=(a_{nk}^{<r>})_{n,k=1}^\infty$ for the matrix with the rows $A_n^{<r>}=0$ for $1\leq n\leq r$ and $A_n^{<r>}=A_n$ for $n\geq r+1$.
(a) Let $L\in(h_d,[c])$ and $A=(a_{nk})_{n,k=1}^\infty$ be the matrix that represents L.

First we show that the limits in (3.9) exist.

Let $x\in h_d$ be given. We write $y_n=A_nx=L_n(x)$ for $n=1,2,\dots$, $\eta(x)$ for the $[c]$–limit of the sequence $y=(y_n)_{n=1}^\infty$ and $\mu_r(x)=\|\mathcal{R}_r(x)\|_{[c_\infty]}$ for all r. Then we have for all r

$$\mu_r(y)=\|\mathcal{R}_r(y)\|_{[c_\infty]}=\sup_{m\geq r+1}\frac{1}{m}\sum_{n=r+1}^{m}|ny_n-(n+1)y_{n-1}-\eta(x)|$$
$$\geq\sup_{m\geq r+2}\frac{1}{m}\sum_{n=r+2}^{m}|ny_n-(n+1)y_{n-1}-\eta(x)|=\|\mathcal{R}_{r+1}(y)\|_{[c_\infty]}=\mu_{r+1}(y),$$

hence $\sup_{x\in Q}\mu_r(y)\geq\sup_{x\in Q}\mu_{r+1}(y)\geq 0$ for all r and for all $Q\in\mathcal{M}_{h_d}$. Consequently

$$\mu(Q)=\lim_{r\to\infty}\mu_r(Q)\text{ exists for all }Q\in\mathcal{M}_{h_d}.$$

Now we define the matrix $B=(b_{nk})_{n,k=1}^\infty$ by $b_{nk}=a_{nk}-\alpha_k$ for all n and k, and denote the unit sphere in h_d by S_{h_d}. Since $\eta(x)=\sum_{k=1}^\infty\alpha_kx_k$ for all $x\in h_d$ by (2.8), it follows that $(\mathcal{R}_r\circ L)(x)=B^{<r>}x$ for all $x\in h_d$, and so by (2.2) and (2.5)

$$\sup_{x\in S_{h_d}}\|\mathcal{R}_r\circ L)(x)\|_{[c_\infty]}=\|B^{<r>}\|_{(h_d,[c_\infty])}$$
$$=\sup_{l,m}\frac{1}{ld_m}\sum_{n=1}^{l}\left|n\sum_{k=1}^{m}b_{nk}^{<r>}-(n-1)\sum_{k=1}^{m}b_{n-1,k}\right|$$
$$=\sup_{m;l\geq r+1}\frac{1}{ld_m}\sum_{n=r+1}^{l}\left|\sum_{k=1}^{m}na_{nk}-(n-1)a_{n-1,k}-\alpha_k\right|.$$

Finally we get (3.9) by (3.6) and by (3.3).

(b) The proof is similar to that of Part (a) with $\alpha_k = 0$ for all k and (3.5) instead of (3.6). □

Finally the characterisations of the classes $\mathcal{K}(h_d, [c])$ and $\mathcal{K}(h_d, [c_0])$ are immediate consequences of (3.4) and Theorem 3.4.

Corollary 3.5 *(a) Let $L \in \mathcal{B}(h_d, [c])$. Then $L \in \mathcal{K}(h_d, [c])$ if and only if*

$$\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{ld_m}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}(na_{nk}-(n-1)a_{n-1,k}-\alpha_k)\right|\right)=0,$$

where the complex numbers α_k are defined in (2.3)*.*

(b) Let $L \in \mathcal{B}(h_d, [c_0])$. Then $L \in \mathcal{K}(h_d, [c_0])$ if and only if

$$\lim_{r\to\infty}\left(\sup_{m;l\geq r}\frac{1}{ld_m}\sum_{n=r}^{l}\left|\sum_{k=1}^{m}na_{nk}-(n-1)a_{n-1,k}\right|\right)=0.$$

We close with an application of our results.

Example 3.6 We consider the Hahn space $h = h_d$, where $d_k = k$ for all $k = 1, 2, \dots$ and the Cesàro matrix $C_1 = A = (a_{nk})_{n,k=1}^{\infty}$ of order 1, where $a_{nk} = 1/n$ for $1 \leq k \leq n$ and $a_{nk} = 0$ for $k > n$ $(n = 1, 2, \dots)$. Then $L_{C_1} \in \mathcal{K}(h, [c_0])$ and $\|L_{C_1}\| = 1$.

Proof. We write

$$\sigma_{lm}=\sum_{n=1}^{m}\left|\sum_{k=1}^{n}na_{nk}-(n-1)a_{n-1,k}\right|,$$

$$\tau_{lm}=\sum_{n=m+1}^{l}\left|\sum_{k=1}^{m}na_{nk}-(n-1)a_{n-1,k}\right|$$

and

$$s_{lm}=\frac{1}{lm}\sum_{n=1}^{l}\left|\sum_{k=1}^{m}na_{nk}-(n-1)a_{n-1,k}\right| \text{ for all } l \text{ and } m.$$

Then $s_{lm} = (1/lm)(\sigma_{lm} + \tau_{lm})$ for all m and l.

We obtain

$$\sigma_{lm}=\sum_{n=1}^{m}\left|\sum_{k=1}^{n}na_{nk}-(n-1)a_{n-1,k}\right|=m,$$

and

$$\tau_{lm}=\begin{cases}0 & (l\leq m)\\ \sum\limits_{n=m+1}^{l}\left|\sum\limits_{k=1}^{m}na_{nk}-(n-1)a_{n-1,k}\right|=0 & (l\geq m+1),\end{cases}$$

hence $s_{lm} = (1/l)$ for all l and m. So $\sup_{lm} s_{lm} = 1$, that is, $L_{C_1} \in \mathcal{B}(h_d, [c_\infty])$ and $\|L_{C_1}\| = 1$ by Theorem 2.4 (a) and (d). Also, for each fixed k,

$$0 \le \frac{1}{l} \sum_{n=1}^{l} |na_{nk} - (n-1)a_{n-1,k}| = \frac{1}{l} \sum_{n=k}^{l} |na_{nk} - (n-1)a_{n-1,k}|) \frac{1}{l} \to 0 \ (l \to \infty),$$

and this and $L_{C_1} \in \mathcal{B}(h_d, [c_\infty])$ togther imply $L_{C_1} \in \mathcal{B}(h_d, [c_0])$ by Theorem 2.4 (c).

Finally, we write for all $l \ge r$, m and r

$$\begin{aligned} s_{lm}^{(r)} &= \frac{1}{lm} \sum_{n=r}^{l} \left| \sum_{k=1}^{m} na_{nk} - (n-1)a_{n-1,k} \right| \\ &= \frac{1}{lm} \left(\sum_{n=r}^{m} \left| \sum_{k=1}^{n} na_{nk} - (n-1)a_{n-1,k} \right| + \sum_{n=m+1}^{l} \left| \sum_{k=1}^{m} na_{nk} - (n-1)a_{n-1,k} \right| \right) \\ &= \frac{1}{lm}(m - r + 1) \le \frac{1}{l}, \end{aligned}$$

hence

$$\sup_{m; l \ge r} s_{lm}^{(r)} \le \frac{1}{r},$$

and so

$$\lim_{r \to \infty} \sup_{m; l \ge r} s_{lm}^{(r)} = 0.$$

Consequently we have $L_{C_1} \in \mathcal{K}(h, [c_0])$ by Corollary 3.5 (b). □

4. Conclusion

The paper adds new results in recent research concerning the studies of bounded linear and compact operators between BK spaces. In particular, the main results are Theorems 2.4, 2.5, 3.4 and Corollary 3.5. Theorem 2.4 establishes the characterisations of the classes $\mathcal{B}(h_d, Y)$ for $Y \in \{[c]_0, [c], [c]_\infty\}$ by necessary and sufficient conditions on the entries of the infinite matrices A that represent these operators; furthermore it contains a formula for the corresponding operator norms. Theorem 2.5 gives a formula for the $[c]$–limit of $L(x)$, when $L \in \mathcal{B}(h_d, [c])$. Theorem 3.4 establishes an identity and an estimate for the Hausdorff measures $\|L\|_\chi$ of $L \in \mathcal{B}(h_d, [c]_0)$ and $L \in \mathcal{B}(h_d, [c])$, respectively, in terms of the entries of the infinite matrices A that represent L. Corollary 3.5 yields the characterisations of the compact operators in the classes $\mathcal{B}(h_d, [c]_0)$ and $L \in \mathcal{B}(h_d, [c])$. Finally, the results of the paper are applied in Example 3.5 to obtain that the operator $C_1 : h_d \to [c]_0$ of the aritmetic means is compact.

Suggestions for further research would be the characterisations of the dual classes (Y, h_d) for $Y \in \{[c]_0, [c], [c]_\infty\}$, and their subclasses of compact matrix operators, and possible extensions of the results, when Y is generalised to $Y_p \in \{[c]_0^p, [c]^p, [c]_\infty^p\}$ $(1 \le p < \infty)$; here the spaces Y_p obtained by replacing the modulus $|\cdot|$ in the definition the sets in Y by $|\cdot|^p$.

Author statement

The author confirms sole responsibility for the writing, interpretation of results and editing of the manuscript.

Conflict of Interest

The author declares no conflict of interest.

References

Boos, J. (2000). *Classical and Modern Methods in Summability.* Oxford University Press, Oxford.

Borwein, D. (1960). On strong and absolute summability. *Proc. Glasgow Math. Assoc.*, 4:122–139.

Das, R. (2017). On the fine spectrum of the lower triangular matrix $B(r; s)$ over the Hahn sequence space. *Kyungpook Math. J.*, 57:441–455.

de Malafosse, B., Malkowsky, E., and Rakočević, V. (2021). *Operators Between Sequence Spaces and Applications.* Springer Nature Singapore, 152 Beach Road, #21 – 01/04 Gateway East, Singapore 18972, Singapore.

Djolović, I. and Malkowsky, E. (2012). On matrix mappings into some strong Cesàro sequence spaces. *Appl. Math. Comput.*, 218:6155–6163.

Djolović, I. and Malkowsky, E. (2013). Banach algebras of matrix transdormations between some sequence spaces related to Λ–strong convergence and boundedness. *Appl. Math. Comput.*, 219:8779–8789.

Goes, G. (1972). Sequences of bounded variation and sequences of Fourier coefficients II. *JMAA*, 39:477–494.

Hahn, H. (1922). Über Folgen linearer Operationen. *Monatsh. Math. Phys.*, 32:3–88.

Jarrah, A. M. and Malkowsky, E. (2002). Linear operators between some matrix domains. *Rend. Circ. di Mat. Palermo*, Ser II 68:641–655.

Jarrah, A. M. and Malkowsky, E. (2003). Ordinary, absolute and strong summability and matrix transformations. *Filomat*, 17:59–78.

Kamthan, P. K. and Gupta, M. (1981). *Sequence Spaces and Series.* Marcel Dekker, New York.

Kirişci, M. (2013a). The Hahn sequence space defined by the Cesàro mean. *Abstr. Appl. Anal.*, 2013. Article ID 817659, 6 pages.

Kirişci, M. (2013b). A survey of the Hahn sequence space. *Gen. Math. Notes*, 19(2):37–58.

Kuttner, B. and Thorpe, B. (1979). Strong convergence. *J. Reine Angew. Math.*, 311/312:42–55.

Malkowsky, E. (1995). The continuous duals of the spaces $c_0(\Lambda)$ and $c(\Lambda)$ for exponentially bounded sequences. *Acta Sci. Math. (Szeged)*, 61:241–250.

Malkowsky, E. (2000). The dual spaces if the sets of Λ–strongly bounded sequences. *Novi Sad J. Math.*, 30(3):99–110.

Malkowsky, E. (2013). Characterization of compact operators between certain BK spaces. *Filomat*, 27(3):447–457.

Malkowsky, E. (2017). On strong summability and convergence. *Filomat*, 31(11):3095–3123.

Malkowsky, E. and Nergiz, H. (2015). Matrix transformations and compact operators on spaces of strongly Cesàro summable and bounded sequences of order α. *CAAM Contemporary Analysis and Applied Mathematics*, 3(263–279).

Malkowsky, E. and Rakočević, V. (1998). The measure of noncompactness of linear operators between certain sequence spaces. *Acta Sci. Math. (Szeged)*, 64:151–170.

Malkowsky, E. and Rakočević, V. (2000). *An Introduction into the Theory of Sequence Spaces and Measures of Noncompactness*, volume 9(17) of *Zbornik radova, Matematički institut SANU*, pages 143–234. Mathematical Institute of SANU, Belgrade.

Malkowsky, E. and Rakočević, V. (2019). *Advanced Functional Analysis.* CRC Press, Taylor & Francis Group, Boca Raton, London, New York.

Malkowsky, E., Rakočević, V., and Tuğ, O. (2021). Compact operators on the Hahn space. *Monatsh. Math.*, pages https://doi.org/10.1007/s00605–021–01588–8.

Mòricz, F. (1989). On Λ–strong convergence of numerical sequences and Fourier series. *Acta MAth. Hung.*, 54:319–327.

Mursaleen, M. and Noman, A. K. (2010). Compactness by the Hausdorff measure of noncompactness. *Nonlinear Anal.*, 73:2541–2557.

Mursaleen, M. and Noman, A. K. (2011). The Hausdorff measure of noncompactness of matrix operators on some BK spaces. *Oper. Matrices*, 5(3):473–486.

Raj, K. and Kiliçman, A. (2014). On generalized difference Hahn sequence spaces. *Hindawi Publishing Corporation, The Scientific World Journal Volume 2014*, 2014. Article ID 398203, 7 pages.

Rao, K. C. (1990). The Hahn sequence space. *Bull. Cal. Math. Soc.*, 82:72–78.

Rao, K. C. and Srinivasalu, T. G. (1996). The Hahn sequence space–II. *Y. Y. U. Journal of Faculty of Education*, 1(2):43–45.

Rao, K. C. and Subramanian, N. (2002). The Hahn sequence space–III. *Bull. Malaysian Math. Sc. Soc. (Second Series)*, 25:163–171.

Ruckle, W. H. (1981). *Sequence Spaces.* Pitman, Boston, London, Melbourne. Research Notes in Mathematics 49.

Toledano, J. M. A., Benavides, T. D., and Acedo, G. L. (1997). *Measures of Noncompactness in Metric Fixed Point Theory*, volume 99 of *Operator Theory Advances and Applications.* Birkhäuser Verlag, Basel, Boston, Berlin.

Wilansky, A. (1984). *Summability through Functional Analysis*, volume 85. North–Holland, Amsterdam. Mathematical Studies.

Zeller, K. and Beekmann, W. (1968). *Theorie der Limitierungsverfahren.* Springer Verlag, Heidelberg, Berlin, New York.