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Lucas Polynomial Solution of a Single Degree of Freedom System

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Abstract

Free vibration of a single degree of freedom system is a fundamental topic in mechanical vibrations. The present study introduces a novel and simple numerical method for the solution of this system in terms of Lucas polynomials in the matrix form. Particular and general solutions of the differential equation can be determined by this method. The method is illustrated by a numerical application and the results obtained are compared with those of the exact solution.

Keywords: Vibration Spring-mass-damper system, Lucas polynomials and series, collocation points and matrix method

1. Introduction

Ordinary differential equations, a crucial part of applied mathe- matics, have many applications in different science and engineering disciplines. The study of the free vibration of damped spring-mass systems having single degree of freedom is fundamental to the understanding of advanced subjects in mechanical vibrations. In many cases, a complicated system can be idealized as a single degree of freedom spring-mass system. Therefore, solving the equations of motion of this system would serve for many other more advanced problems. There are various useful methods for calculating solutions of a spring-mass-damper system excited by a harmonic force. Kurt and Çevik (2008) and Savaşaneril (2018) proposed a matrix method for solving this problem.

The present study introduces a novel and simple method in terms of Lucas polynomials in the matrix form. Lucas polynomials have been used by many researchers for the solution of differential and integral equations. Gümgüm et al. (2018) proposed a Lucas expansion approach for functional integro-differential equations involving variable delays. Baykus (2017) used this method to find the approximate solution of high-order pantograph type delay differential equations with variables delays. Gümgüm et al. (2020) gave an approach for Second Order Nonlinear Differential Equations via Lucas polinomial. The method has also been used to solve nonlinear equations by Gümgüm et al. (2019). Yüzbaşı and Yıldırım (2020) proposed Pell-Lucas collocation method to solve high-order linear Fredholm-Volterra integro-differential equations. In addition, Kübra et al. (2021) , Yüzbaşı and Ismailov (2018), Kürkçü and Sezer (2022) proposed a different the matrix method.

In general, an m^{th} order differential equation can be written as:

$$\sum_{k=0}^{m} P_k x^k(t) = f(t), \tag{1}$$

with initial conditions

$$\sum_{k=0}^{m-1} a_{ik} x^{(k)}(a) = \lambda_i, \quad i = 0,1,...,m-1, \tag{2}$$

where $P_k(t)$ are analytic functions defined on $a \leq t \leq b$, and a_{ik}, λ_i are suitable constants. In the present method, the solution of Eq. (1) is expressed in the Lucas polynomial form as:

$$x(t) = x_N(t) = \sum_{n=0}^{N} a_n L_n(t), \tag{3}$$

where $L_n(t)$ are the Lucas polynomials and a_n, $n=0,1,2,\ldots N$ are unknown coefficients (Baykus, N., 2017).

2. Fundamental Matrix Relations

In this section, we constitute the matrix forms of the unknown function $x(t)$ defined by Eq. (3) and the derivative $x^{(k)}(t)$ in Eq. (1). We can first write the truncated Lucas series (3) in the matrix form, for $n=0,1,2,\ldots.N$:

$$x(t) = x_N(t) = \mathbf{L}(t)\mathbf{A}, \tag{4}$$

where

$$\mathbf{L}(t) = \begin{bmatrix} L_0(t) & L_1(t) & \cdots & L_N(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T \tag{5}$$

$$\begin{aligned} &L_0(t) = 2, \\ &L_1(t) = t, \\ &\vdots \\ &L_{n+1}(t) = t\,L_n(t) + L_{n-1}(t), \quad n \geq 1. \end{aligned} \tag{6}$$

Then, by using the Lucas polynomials $L_n(t)$ given by Eq. (6), we write the matrix form $\mathbf{L}(t)$ as follows:

$$\mathbf{L}(t) = \mathbf{T}(t)\mathbf{M}, \tag{7}$$

where

$$\mathbf{T} = \begin{bmatrix} 1 & t & \cdots & t^N \end{bmatrix}. \tag{8}$$

If N is odd,

$$\mathbf{M}^T = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{(n-1)}{\left(\frac{n-1}{2}\right)}\binom{\frac{n-1}{2}}{\frac{n-1}{2}} & 0 & \frac{(n-1)}{\left(\frac{n+1}{2}\right)}\binom{\frac{n+1}{2}}{\frac{n-3}{2}} & \cdots & 0 \\ 0 & \frac{n}{\left(\frac{n+1}{2}\right)}\binom{\frac{n+1}{2}}{\frac{n-1}{2}} & 0 & \cdots & \frac{n}{n}\binom{n}{0} \end{bmatrix}. \tag{9}$$

If N is even,

$$\mathbf{M}^T = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \frac{(n-1)}{\left(\frac{n}{2}\right)}\binom{\frac{n}{2}}{\frac{n-2}{2}} & 0 & \cdots & 0 \\ \frac{n}{\frac{n}{2}}\binom{\frac{n}{2}}{\frac{n}{2}} & 0 & \frac{n}{\left(\frac{n+2}{2}\right)}\binom{\frac{n+2}{2}}{\frac{n-2}{2}} & \cdots & \frac{n}{n}\binom{n}{0} \end{bmatrix}. \tag{10}$$

By the matrix relations in Eq. (4) and Eq. (7), it follows that,

$$x_N(t) = \mathbf{T}(t)\mathbf{MA}. \tag{11}$$

Besides, it is well known from Gümgüm et al. (2019) that the relation between **T**(t) and its derivative $\mathrm{T}^{(k)}(t)$ is of the form:

$$\mathbf{T}^{(k)}(t)=\mathbf{T}(t)\,\mathbf{B}^{k}, \tag{12}$$

where

$$\mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{13}$$

and $\mathbf{B}^{0}$ is a unit matrix.

By using Eq. (11) and Eq. (12), we have the matrix relation:

$$x_N^{(k)}(t)=\mathbf{T}(t)\,\mathbf{B}^{k}\mathbf{M}\,\mathbf{A},\quad k=0,1,...,m. \tag{14}$$

Inserting the collocation points

$$x_i=a+\frac{b-a}{N}i,\quad i=0,1,...,N \tag{15}$$

into Eq. (1) gives

$$\sum_{k=0}^{m}P_k x^k(t_i)=f(t_i), \tag{16}$$

which can be written in matrix form as:

$$\mathbf{W}=\left[w_{pq}\right]=\sum_{k=0}^{m}P_k\mathbf{T}(t)\,\mathbf{B}^{k}\mathbf{M},\quad p,q=0,1,...,N.$$

Now, by the relation Eq. (14), we can obtain the condition matrix form for the initial conditions (Eq. (2)) $\mathbf{U}_i\mathbf{A}=\lambda$ or

$$\left[\mathbf{U}_i \quad ; \quad \lambda_i\right],\quad i=0,1,...,m-1 \tag{17}$$

such that

$$\mathbf{U}_i=\sum_{k=0}^{m-1}a_{ik}T(a)\mathbf{B}^{k}\mathbf{M}=\left[u_{i0}\quad u_{i1}\quad\cdots\quad u_{iN}\right]. \tag{18}$$

In order to determine the particular solution of the problem in matrix form, Eq. (17) is written briefly in the form:

$$\mathbf{WX}=\mathbf{F}\qquad \text{or}\qquad \left[\mathbf{W}\quad ;\quad \mathbf{F}\right], \tag{19}$$

where

$$\mathbf{W}=\left[w_{pq}\right]=\sum_{k=0}^{m}P_k\mathbf{T}(t)\,\mathbf{B}^k\mathbf{M},\quad p,q=0,1,...,N. \tag{20}$$

By consequence,

$$\mathbf{X}=\mathbf{W}^{-1}\mathbf{F}, \tag{21}$$

which yields the desired Lucas cofficients x_n, $n=0,1,2,\ldots,N$ of the particular solution.

Now, to solve the problem, the following augmented matrix is constructed by replacing the last 2 rows of [**W** ; **F**] of Eq. (22) by the 2-row matrix [$\mathbf{U}_i$; λ_i]:

$$\left[\mathbf{W};\mathbf{F}\right]=\begin{bmatrix} w_{00} & w_{01} & & w_{0N} & ; & f_0(t) \\ w_{10} & w_{11} & & w_{1N} & ; & f_1(t) \\ \vdots & \vdots & & \vdots & \vdots & \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & f_{N-m}(t) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}. \tag{22}$$

3. Lucas Matrix – Collocation Technique of The Problem

In this study, the viscously damped single degree of freedom system subjected to harmonic excitation (Inman, 2001):

$$M\,\ddot{x}+C\,\dot{x}+Kx=F_0\cos wt, \tag{23}$$

with initial conditions

$$\begin{aligned} x(0)&=\lambda_0 \\ \dot{x}(0)&=\lambda_1 \end{aligned} \tag{24}$$

will be solved by Lucas matrix method. In this case, we have m=2 and the constants $P_2=M$, $P_1=C$, $P_0=K$ *in* Eq. (1):

$$M\,\ddot{x}+C\,\dot{x}+Kx=\sum_{k=0}^{2}P_k\mathbf{T}(t)\,\mathbf{B}^k\mathbf{M}\,\mathbf{X}. \tag{25}$$

3.1. Particular solution

In order to determine the particular solution of the problem in matrix form, Eq. (23) is written briefly in the form:

$$\mathbf{W}=\left[w_{pq}\right]=\sum_{k=0}^{2}P_k\mathbf{T}(t)\,\mathbf{B}^k\mathbf{M},\quad p,q=0,1,...,N\ . \tag{26}$$

By consequence,

$$\mathbf{X} = \mathbf{W}^{-1}\mathbf{F}, \tag{27}$$

which yields the desired Lucas cofficients x_n, $n=0,1,2,\ldots,N$ of the particular solution.

3.2. General solution

To determine the general solution, the matrix form of the boundary conditions (21) is written as:

$$\mathbf{U}_i = \sum_{k=0}^{1} a_{ik} T(a)\mathbf{B}^k\mathbf{M} = \begin{bmatrix} u_{i0} & u_{i1} & \cdots & u_{iN} \end{bmatrix} \qquad i = 0,1 \tag{28}$$

Now, to solve the problem, the following augmented matrix is constructed by replacing the last 2 rows of [**W** ; **F**] of Eq. (26) by the 2-row matrix [$\mathbf{U}_i$; λ_i]:

$$\left[\mathbf{W};\mathbf{F}\right] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f_0 \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f_1 \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{N-2,0} & w_{N-2,1} & \cdots & w_{N-2,N} & ; & f_{N-2} \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \end{bmatrix}. \tag{29}$$

In Eq. (25), if $rank\tilde{\mathbf{W}} = rank\left[\tilde{\mathbf{W}};\tilde{\mathbf{F}}\right] = N+1$, then the coefficient matrix A is uniquely determined and so the solution of the problem Eq. (1)-(2) is obtained as:

$$x_N(t) = \mathbf{L}(t)\mathbf{A} \text{ or } x_N(t) = \mathbf{T}(t)\mathbf{M}\mathbf{A}. \tag{30}$$

4. Numerical Example

A spring-mass-damper system with a mass of M = 10 kg, damping coefficient of C = 20 kg/s and spring stiffness of K = 4000 N/m subject to an excitation force of amplitude $F_0=100N$ and frequency $\omega=10$ rad/s is considered with initial conditions $x(0) = 0.01, \dot{x}(0) = 0$. The matrix operations in this section are performed by using Wolfram Mathematica 13.0.

The exact solution is given by (Inman, D.J., 2001):

$$x(t) = \tan^{-1}\frac{x_0\omega_d}{v_0 + \xi\omega_n x_0}, \qquad A = \frac{1}{\omega_d}\sqrt{(v_0+\xi\omega_n x_0)^2 + (x_0\omega_d)^2}, \qquad x = 0,$$

for free response, and,

$$x(t) = \tan^{-1}\frac{\omega_d(x_0 - x\cos\theta)}{v_0 + (x_0 - x\cos\theta)\xi\omega_n - \omega x\sin\theta}, \qquad A = \frac{x_0 - x\cos\theta}{\sin\phi},$$

$$\theta = \tan^{-1}\frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2}, \qquad x = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}},$$

for forced response. From the (11) fundamental matrix equation is:

$$10\mathbf{T}(t).\mathbf{B}^2\mathbf{M}^T\mathbf{A} + 20\mathbf{T}(t).\mathbf{B}\mathbf{M}^T\mathbf{A} + 4000\mathbf{T}(t)\mathbf{M}^T\mathbf{A} = 100\cos 10t. \tag{31}$$

By using (15) the collocation points for N = 5 is calculated as:

$$\left\{0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1\right\},$$

and we obtain:

$$\mathbf{W}=\begin{bmatrix} 8000 & 20 & 8020 & 60 & 8080 & 100 \\ 8000 & 820 & 8188 & 2506.4 & 8763.84 & 4335.04 \\ 8000 & 1620 & 8676 & 5149.6 & 10830.7 & 9604.32 \\ 8000 & 2420 & 9484 & 8181.6 & 14514.9 & 17075.2 \\ 8000 & 3220 & 10612 & 11794.4 & 20204.2 & 28226.1 \\ 8000 & 4020 & 12060 & 16180 & 28440 & 45000 \end{bmatrix}.$$

The augmented matrix for this fundamental matrix equation is calculated as:

$$\left[\mathbf{W};\mathbf{G}\right]=\begin{bmatrix} 8000 & 20 & 8020 & 60 & 8080 & 100 & ; & 100 \\ 8000 & 820 & 8188 & 2506.4 & 8763.84 & 4335.04 & ; & 28.3662 \\ 8000 & 1620 & 8676 & 5149.6 & 10830.7 & 9604.32 & ; & -65.3644 \\ 8000 & 2420 & 9484 & 8181.6 & 14514.9 & 17075.2 & ; & 96.017 \\ 2 & 0 & 2 & 0 & 2 & 0 & ; & 0.01 \\ 0 & 1 & 0 & 3 & 0 & 5 & ; & 0 \end{bmatrix}.$$

Performing the necessary matrix operations, the particular solution is determined as:

$$x_p(t)=184063-160037L_1(t)-123718L_2(t)+88047.9L_3(t)+31679L_4(t)-20838L_5(t)$$

in Lucas polynomial form, and the general solution is:

$$x_g(t)=323.256-313.714L_1(t)-216.497L_2(t)+168.799L_3(t)+54.8743L_4(t)-38.5364L_5(t)$$

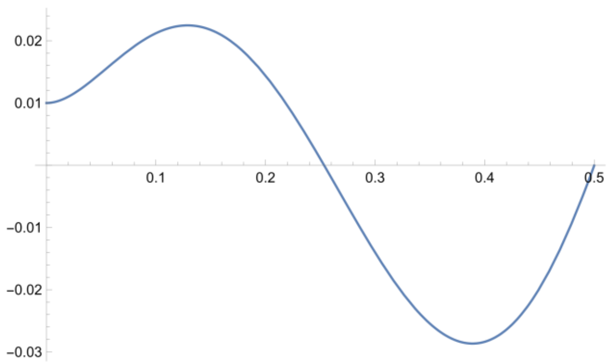

Figure 1. The general solution

Finally $x_{40}(t)$ is obtained for N=40:

$$
\begin{aligned}
x_{40}(t) &= -5.894703961761931.10^7 + 4.809657176756848.10^8.L_1(t) + 7.467470466915241.10^7.L_2(t) \\
&-4.293391442061826.10^8.L_3(t) - 8.93847819837741.10^8.L_4(t) + 3.132395109718503.10^8.L_5(t) + \\
&6.567014129734314.10^7.L_6(t) - 1.59204644360381.10^8.L_7(t) - 2.54515240071604.10^7.L_8(t) \\
&+3.495715330499887.L_9(t) + 6614643.904937126.L_{10}(t) + 1.748782280616832.10^7.L_{11}(t) \\
&-7920874.169940725.L_{12}(t) - 1.725364834634807.10^7.L_{13}(t) + 9165009.245850491.L_{14}(t) \\
&5109995.376942811.L_{15}(t) - 5203828.325874632.L_{16}(t) + 220814.41721369085.L_{17}(t) \\
&+1459939.562524835.L_{18}(t) - 569693.2886912221.L_{19}(t) - 112126.0439566288.L_{20}(t) \\
&+167623.56494754046.L_{21}(t) - 43435.56898627258.L_{22}(t) - 17906.61385994068.L_{23}(t) \\
&-10833.4183322996338..L_{24}(t) - 3193.338078951663.L_{25}(t) + 25413.043724168358.L_{26}(t) \\
&-1638.6461992483767.L_{27}(t) - 10040.921604866502.L_{28}(t) + 1137.8678286578668.L_{29}(t) \\
&1236.9842049196193.L_{30}(t) + 3300786570862636.L_{31}(t) - 174.6007432535557.L_{32}(t) \\
&-65.0753966189064.L_{33}(t) + 35.56471110532434.L_{34}(t) - 26.896999946160374.L_{35}(t) \\
&7.621730624316185.L_{36}(t) - 3.0780533665079806.L_{37}(t) + 6.4648253275926315.L_{38}(t) \\
&-3.293481976937444.L_{39}(t) + 0.4951794203008488.L_{40}(t)
\end{aligned}
$$

The results for different values of N are compared in Table 1. In the table, it is obviously seen that the Lucas solution approaches the exact solution as the truncation limit N is increased.

Table 1. Convergence of the Lucas results to those of exact solution

t	Exact x(t)	Present Method (N =5) x_5(t)	Present Method (N =30) x_{30}(t)	Present Method (N = 40) x40 (t)
0	0.010000000102	0.010000000000033538	0.0099998963996768	0.010000037786085159
0.1	0.026606378588454577	0.021218558662097035	0.026608246698293588	0.02660616339402371
02	0.002080749042409633	0.014399161349069003	0.00207760236634558	0.0020807331586712163
0.3	−0.04851575510425815	−0.01401641114458152	−0.04851778585172134	−0.048516383083797576
0.4	−0.022765129712940937	−0.02837548614445584	−0.022765300714933545	−0.02276530628618173
0.5	0.019920584607770103	−0.00005820215181184807	0.01991818146927926	0.019921048858392965
0.6	0.021295471543613383	0.04627878810085573	0.021294628886607825	0.0212938860227041
0.7	0.023585587433570195	−0.013541385405768658	0.023586714873090386	0.023585824521433096
0.8	0.007669134885655218	−0.4494595130334105	0.007668879581615329	0.0076684916857630014
0.9	−0.03468236386751298	−1.723424008541791	−0.034679219126701355	−0.034682015888392925
1	−0.033476404064609305	−4.535634612144179	−0.03263649344444275	−0.03311382979154587

As N increased, better results were obtained. The efficiency and validity of the method are obvious.

5. Conclusions

A Lucas polynomial matrix solution has been presented for the periodic motion of an underdamped single degree of freedom spring-mass system subjected to harmonic excitation. Both particular and general solutions of the differential equation can be determined by this method. The results show a very good agreement with the method of undetermined coefficients (exact solution). The solution can also be applied to higher order systems with the same simplicity and application of the method to these problems offers a considerable facility. The method is applicable to any function that can be expanded to Lucas series. Accurate results can be obtained with rather low values of the truncation limit N; however, in order to have a better approximation, the truncation limit should be increased.

Author Statement

The author confirms sole responsibility for the following: study conception and design, data collection, analysis and interpretation of results, and manuscript preparation.

Conflict of Interest

The authors declare no conflict of interest.

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