Scientific Research Communications

Solution of the Single Degree of Freedom System Using Bernoulli Collocation Method

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Research Article

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Received: 6 June 2024
Accepted: 8 January 2025
Published: 24 January 2025

Keywords
Vibration
Spring-mass-damper system
Bernoulli polynomials
Collocation points
Approximate methods

Abstract

The fundamental subject of a single degree of freedom system's free vibration is essential in the field of mechanical vibrations, with applications in a wide range of engineering fields. This paper presents a novel numerical method for solving this problem based on Bernoulli polynomials in matrix form. The method is simple to implement and requires only basic linear algebra operations. The method is also very efficient; and can be used to solve problems with the single degree of freedom system. The proposed method is illustrated by a numerical example, and the results are compared with those of the exact solution. The results show that the proposed method is highly accurate and efficient.

1. Introduction

Understanding the natural oscillations of damped spring-mass systems with a single degree of freedom is a cornerstone of mechanical vibrations. In mechanical vibrations, the free vibration of damped spring-mass systems having single degree of freedom is essential for advanced systems. Interestingly, many complex systems can be simplified to a single degree of freedom spring-mass model. As a result, solving the equations of motion for this system can be used to solve many other more complex problems. There are a variety of effective methods for calculating solutions for a spring-mass-damper system excited by a harmonic force.

The approximation method based on Taylor polynomials (Kurt and Çevik, 2008), Laguerre (Savaşaneril, 2018) and Lucas polynomials (Savaşaneril, 2023) is used for the solution of single degree of freedom system. Also, an exponential matrix method is developed to solve delayed single degree of freedom system (Çevik et al., 2014). In addition, different matrix methods based on different polynomials have been developed for various type systems of differential equations (Yüzbaşı and Karaçayır, 2017; Yüzbaşı et al., 2012; Yüzbaşı and Yıldırım, 2021; Sezer and Kaynak, 1996; Bahşi et al., 2018; Gülsu et al., 2011; Baykuş and Sezer, 2017; Biçer and Dağ, 2023; Kürkçü et al., 2016; Çayan et al., 2022; Yıldız and Sezer, 2019).

How to cite this paper: Erdem Biçer, K. (2025). Solution of the single degree of freedom system using Bernoulli collocation method. *Scientific Research Communications*, *5*(1), 1–10. https://doi.org/10.52460/src.2025.001

In this study, a matrix method based on Bernoulli polynomial is used to solve the single degree of freedom system. Also, this method has been used to solve high-order linear differential-difference equations, linear delay difference equations with variable coefficients, mixed linear Fredholm integro-differential-difference equations, nonlinear differential equations (Erdem and Yalçınbaş, 2012a; Erdem and Yalçınbaş, 2012b; Erdem et al., 2013; Erdem Biçer and Sezer, 2019).

We consider the Bernoulli polynomial solution of an m^{th} order differential equation given in Eq. (1):

$$\sum_{k=0}^{m} P_k x^{(k)}(t) = f(t), \tag{1}$$

with initial conditions:

$$\sum_{k=0}^{m-1} a_{ik} x^{(k)}(a) = \lambda_i,\ i = 0,1,\cdots,m-1. \tag{2}$$

$P(t)$ is analytic function defined on $a \leq t \leq b$ a_{ik}, λ_i are suitable constants. In the present method, the solution of (1) is expressed in the Bernoulli polynomial form as in Eq. (3):

$$x(t) = x_N(t) = \sum_{n=0}^{N} a_n B_n(t), \tag{3}$$

where, $B_n(t)$ is the Bernoulli polynomials and a_n, $n = 0,1,\cdots,N$ are unknown coefficients (Gülsu et al., 2011).

2. Fundamental Matrix Relations

The approximate solution of Eq. (1) in terms of Bernoulli polynomials can be expressed as in Eq. (4):

$$x(t) = x_N(t) = \mathbf{B}(t)\mathbf{A} \tag{4}$$

where,

$$\mathbf{B}(t) = [B_0(t) \quad B_1(t) \quad \cdots \quad B_N(t)], \mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T. \tag{5}$$

Here, the Bernoulli polynomials $B_n(x)$ is defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n$$

or

$$B_n(x) = \sum_{r=0}^{n} \binom{n}{r} b_r x^{n-r},\ \text{which } b_r = B_r(0) \text{ (Apostol, 1976)}.$$

The first few Bernoulli polynomials with respect to t follows in Eq. (6):

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= t - \frac{1}{2}, \\ B_2(x) &= t^2 - t + \frac{1}{6}, \\ B_3(x) &= t^3 - \frac{3}{2}t^2 - \frac{1}{2}t, \\ &\vdots \end{aligned} \tag{6}$$

Then, by using the Bernoulli polynomials $B_n(t)$given by (6), we write the matrix form $\mathbf{B}(t)$ as follows in Eq. (7):

$$\mathbf{B}(t) = \mathbf{T}(t)\boldsymbol{\zeta}^T. \tag{7}$$

where,

$$\mathbf{T} = [1 \quad \mathrm{t} \quad \cdots \quad t^N]. \tag{8}$$

$$\boldsymbol{\zeta} = \begin{bmatrix} \binom{0}{0}b_0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{1}{0}b_1 & \binom{1}{1}b_0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{2}{0}b_2 & \binom{2}{1}b_1 & \binom{2}{2}b_0 & 0 & 0 & \cdots & 0 \\ \binom{3}{0}b_3 & \binom{3}{1}b_2 & \binom{3}{2}b_1 & \binom{3}{3}b_0 & 0 & \cdots & 0 \\ \binom{4}{0}b_4 & \binom{4}{1}b_3 & \binom{4}{2}b_2 & \binom{4}{3}b_1 & \binom{4}{4}b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{N}{0}b_N & \binom{N}{1}b_{N-1} & \binom{N}{2}b_{N-2} & \binom{N}{3}b_{N-3} & \binom{N}{4}b_{N-4} & \cdots & \binom{N}{N}b_0 \end{bmatrix}. \tag{9}$$

Using the matrix relations in Eq. (4) and Eq. (7), it follows that:

$$x_N(t) = \mathbf{T}(t)\boldsymbol{\zeta}^T\mathbf{A}. \tag{10}$$

For the matrix relation between $\mathbf{T}(t)$ and its k −th derivative $\mathbf{T}^{(k)}(t)$ as:

$$\mathbf{T}^{(k)}(t) = \mathbf{T}(t)\mathbf{E}^k. \tag{11}$$

where,

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{12}$$

By substituting the matrix form Eq. (11) into Eq. (10), the following matrix relation is obtained:

$$x_N^{(k)}(t) = \mathbf{T}(t)\mathbf{E}^k\boldsymbol{\zeta}^T\mathbf{A}, \qquad k = 0,1,\cdots,m. \tag{13}$$

3. Bernoulli Collocation Method

By using the collocation points,

$$x_i = a + \frac{b-a}{N} i, \qquad i = 0,1,\cdots,N. \tag{14}$$

into Eq. (1) gives Eq. (15):

$$\sum_{k=0}^{m} P_k x^{(k)}(t_i) = f(t_i), \tag{15}$$

which can be written in matrix form as:

$$\boldsymbol{W} = [w_{pq}] = \sum_{k=0}^{m} P_k \boldsymbol{T}(t) \boldsymbol{E}^k \boldsymbol{\zeta}^T, p, q = 0,1,\cdots,N.$$

For the particular solution, Eq. (15) is written briefly as Eq. (16):

$$\mathbf{WX} = \mathbf{F} \text{ or } [\mathbf{W}; \mathbf{F}] \tag{16}$$

where,

$$\mathbf{W} = [w_{pq}] = \sum_{k=0}^{m} P_k \mathbf{T}(t) \mathbf{E}^k \boldsymbol{\zeta}^T, p, q = 0,1,\cdots,N. \tag{17}$$

Therefore, unknown Bernoulli coefficients matrix is obtained as in Eq. (18):

$$\mathbf{X} = \mathbf{W}^{-1}\mathbf{F} \tag{18}$$

which yields the desired Bernoulli coefficients $x_n, n = 0,1,\cdots,N$ of the particular solution. The matrix form of the initial conditions in Eq. (2) is obtained as in Eq. (19):

$$\mathbf{U}_{\boldsymbol{i}}\mathbf{A} = \lambda \text{ or } [\mathbf{U}_{\boldsymbol{i}}; \lambda_i], i = 0,1,\cdots,m-1 \tag{19}$$

where,

$$\mathbf{U}_{\boldsymbol{i}} = \sum_{k=0}^{m-1} a_{ik} \mathrm{T}(a) \mathbf{E}^k \boldsymbol{\zeta}^T = [u_{i0} \quad u_{i1} \quad \cdots \quad u_{iN}]. \tag{20}$$

Now, to solve the problem, the following augmented matrix is constructed by replacing the last 2 rows of $[\mathbf{W}; \mathbf{F}]$ by the 2-row matrix $[\mathbf{U}_{\boldsymbol{i}}; \lambda_i]$:

$$[\mathbf{W}; \mathbf{F}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f_0(t) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f_1(t) \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & f_{N-m}(t) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}. \tag{21}$$

4. Solution of the Problem with Bernoulli Collocation Method

In this study, the solution of the viscously damped single degree of freedom system subjected to harmonic excitation (Inman, 2001):

$$M\ddot{x} + C\dot{x} + Kx = F_0 coswt \tag{22}$$

with initial conditions,

$$\begin{aligned} x(0) &= \lambda_0 \\ \dot{x}(0) &= \lambda_1 \end{aligned} \tag{23}$$

will be examined. Eq. (22) is a second order differential equation so $m = 2$ in Eq (1). Also, the constants are

$$P_2 = M, P_1 = C, P_0 = K.$$

5. Particular Solution

For the particular solution of the problem in matrix form, Eq. (22) is written briefly in the form of Eq. (24):

$$M\ddot{x} + C\dot{x} + Kx = \sum_{k=0}^{2} P_k \mathbf{T}(t)\mathbf{E}^k \boldsymbol{\zeta}^T \mathbf{X} = \mathbf{F} \tag{24}$$

or shortly,

$$\mathbf{WX} = \mathbf{F} \text{ or } [\mathbf{W}; \mathbf{F}]$$

where,

$$\mathbf{W} = \left[w_{pq}\right] = \sum_{k=0}^{2} P_k \mathbf{T}(t)\mathbf{E}^k \boldsymbol{\zeta}^T, p, q = 0,1, \cdots, N, \tag{25}$$

So, the unknown Bernoulli coefficients $x_n, n = 0,1, \cdots, N$ are obtained as in Eq. (26):

$$\mathbf{X} = \mathbf{W}^{-1}\mathbf{F} \tag{26}$$

for the particular solution.

6. General Solution

For the general solution of the problem, the matrix form of the initial conditions Eq. (23) is written as in Eq. (27)

$$\mathbf{U}_i = \sum_{k=0}^{1} a_{ik} \mathrm{T}(a)\mathbf{E}^k \boldsymbol{\zeta}^T = [u_{i0} \quad u_{i1} \quad \cdots \quad u_{iN}]\ i = 0,1. \tag{27}$$

and the augmented matrix is obtained as in Eq. (28)

$$[\mathbf{W};\mathbf{F}]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f_0(t) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f_1(t) \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ w_{N-2,0} & w_{N-2,1} & \cdots & w_{N-2,N} & ; & f_{N-2}(t) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \end{bmatrix}. \tag{28}$$

In Eq. (24), if $rank\mathbf{W}=rank[\mathbf{W};\mathbf{F}]=N+1$ then the coefficient matrix A is uniquely determined and so the solution of the problem (1-2) is obtained as in Eq. (29):

$$x_N(t)=\mathbf{B}(t)\mathbf{A} \text{ or } x_N(t)=\mathbf{T}(t)\boldsymbol{\zeta}^T\mathbf{A}. \tag{29}$$

7. Numerical Example

A spring-mass-damper system with a mass of $M = 10$ kg, damping coefficient of $C = 20\ kg/s$ and spring stiffness of $K = 4000\ N/m$ subject to an excitation force of amplitude $F_0 = 100\ N$ and frequency $\omega = 10\ rad/s$ rad/s is examined with initial conditions $x(0) = 0.01, \dot{x}(0) = 0$. Based on these parameters, Eq. (22) can be expressed as

$$10\ddot{x}+20\dot{x}+4000x=100\ cos10t.$$

The exact solution obtained by the method of undetermined coefficients is as follows (Inman, 2001)

$$x(t)=tan^{-1}\frac{x_0\omega_d}{v_0+\xi\omega_n x_0}, A=\frac{1}{\omega_d}\sqrt{(v_0+\xi\omega_n x_0)^2+(x_0\omega_d)^2}, x=0 \text{ for free response,}$$

and

$$\begin{aligned} x(t)&=tan^{-1}\frac{\omega_d(x_0-xcos\theta)}{v_0+(x_0-xcos\theta)\xi\omega_n-\omega x sin\theta}, A=\frac{x_0-xcos\theta}{sin\phi} \\ \theta&=tan^{-1}\frac{2\xi\omega_n\omega}{\omega_n^2-\omega^2}, x=\frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\xi\omega_n\omega)^2}} \end{aligned} \quad \text{for forced response.}$$

Firstly, using Eq. (17) the fundamental matrix equations is written as Eq. (30):

$$10\mathbf{T}(t)\mathbf{E}^2\boldsymbol{\zeta}^T\mathbf{A}+20\mathbf{T}(t)\mathbf{E}\boldsymbol{\zeta}^T\mathbf{A}+4000\mathbf{T}(t)\boldsymbol{\zeta}^T\mathbf{A}=100cos10t \tag{30}$$

First, let us determine both the particular solution and the general solution for the equation when $N = 5$. To accomplish this, we need to calculate the necessary matrix equations. For $N = 5$, the matrix $\mathbf{W}$ and the augmented matrix $[\mathbf{W};\mathbf{F}]$ are defined as follows:

$$\mathbf{W}=\begin{bmatrix} 4000 & -1980 & \frac{2000}{3} & -20 & -\frac{340}{3} & -\frac{10}{3} \\ 4000 & -1180 & \frac{104}{3} & \frac{872}{5} & -\frac{1972}{75} & -\frac{6442}{75} \\ 4000 & -380 & -\frac{832}{3} & \frac{428}{5} & \frac{6764}{75} & -\frac{3586}{75} \\ 4000 & 420 & -\frac{808}{3} & -\frac{472}{5} & \frac{6576}{75} & \frac{158}{3} \\ 4000 & 1220 & \frac{176}{3} & -\frac{868}{5} & -\frac{2548}{75} & \frac{6326}{75} \\ 4000 & 2020 & \frac{2120}{3} & 40 & -\frac{340}{3} & -\frac{10}{3} \end{bmatrix},$$

$$[\mathbf{W};\mathbf{F}]=\begin{bmatrix} 4000 & -1980 & \frac{2000}{3} & -20 & -\frac{340}{3} & -\frac{10}{3} & ; & 100 \\ 4000 & -1180 & \frac{104}{3} & \frac{872}{5} & -\frac{1972}{75} & -\frac{6442}{75} & ; & -\frac{4536}{109} \\ 4000 & -380 & -\frac{832}{3} & \frac{428}{5} & \frac{6764}{75} & -\frac{3586}{75} & ; & -\frac{8432}{129} \\ 4000 & 420 & -\frac{808}{3} & -\frac{472}{5} & \frac{6576}{75} & \frac{158}{3} & ; & \frac{5665}{59} \\ 1 & -\frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{30} & 0 & ; & 0.01 \\ 0 & 1 & -1 & \frac{1}{2} & 0 & -\frac{1}{6} & ; & 0 \end{bmatrix}$$

As indicated in the method, the particular solution is obtained as follows after certain matrix operations

$$x_p(t)=8.4282770393x^5-20.4809690511x^4+16.234587123x^3 \\ -4.34431040004x^2+0.0647716655134x+0.0463976936726$$

Similarly, the general solution is as follows:

$$x_g(t)=-61.2945777462x^5+79.2335823084x^4-30.1164823672x^3+3x^2 \\ +1.18423789293\times 10^{-15}x+0.01$$

The problem has also been solved for $N=50$, $N=100$, and $N=200$. To demonstrate the accuracy of the method, the exact solution has been compared with $x_{50}(t)$ in Figure 1, with $x_{100}(t)$ in Figure 2, and with $x_{200}(t)$ in Figure 3. The graphs demonstrate the accuracy and reliability of the proposed method, as the solutions closely correspond to the expected theoretical behavior.

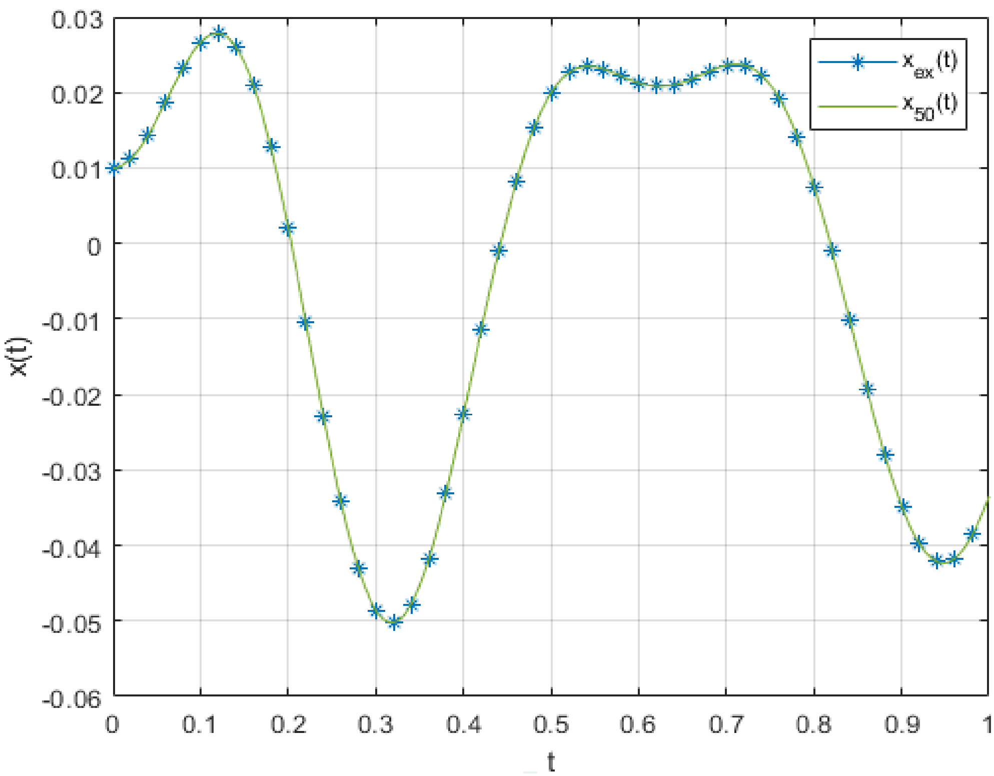

Figure 1. Comparison of the exact method and the approximate numerical method for the general solution of the problem for $N=50$

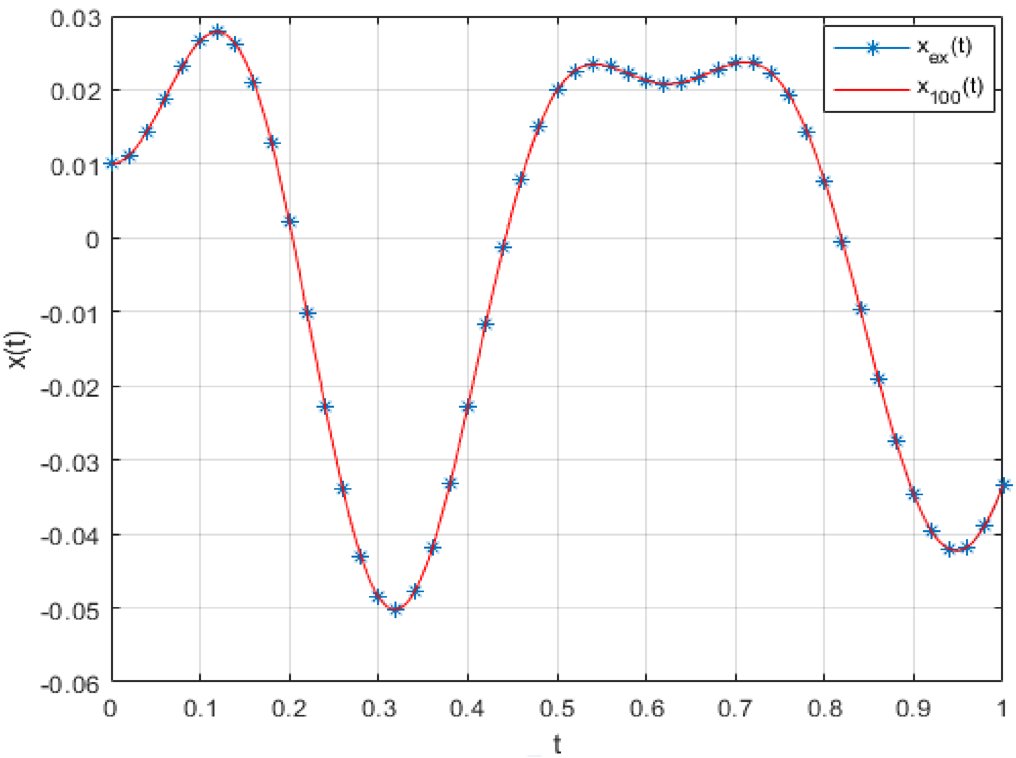

Figure 2. Comparison of the exact method and the approximate numerical method for the general solution of the problem for $N = 100$

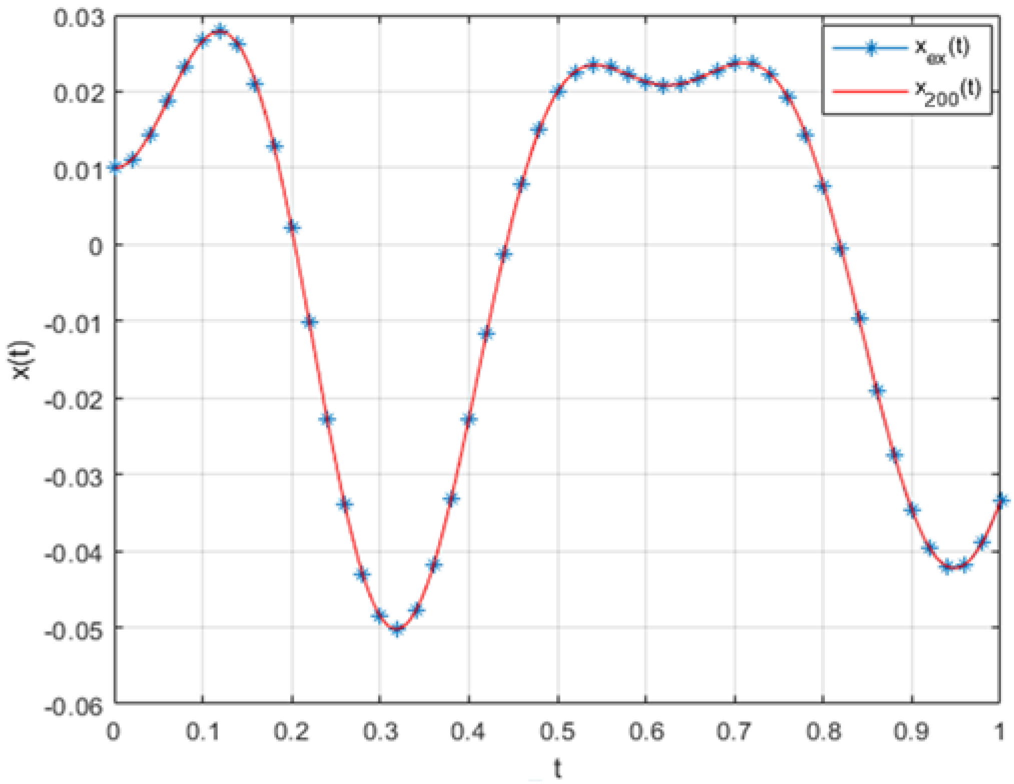

Figure 3. Comparison of the exact method and the approximate numerical method for the general solution of the problem for $N = 200$

8. Conclusions

In this study, Bernoulli polynomials are utilized to develop a novel method for solving the periodic motion of an underdamped single-degree-of-freedom spring-mass system subjected to harmonic excitation. This method is based on collocation points and matrix equations, offering a structured and computationally efficient approach. The problem is solved for different values of the system parameters, and the approximate solutions are illustrated graphically, demonstrating the method's practical applicability.

The results obtained through Matlab calculations highlight the accuracy and reliability of the proposed method, as the solutions closely align with expected theoretical behavior. Moreover, the method's ease of implementation and computational efficiency make it highly advantageous. Specifically, Bernoulli polynomials enable a reduction in computational complexity and provide high accuracy with fewer collocation points.

Beyond the specific application addressed in this study, the proposed approach has broader potential. It can be extended to solve higher-order systems of differential equations and other complex engineering or physical problems that exhibit periodic behavior. For instance, systems in vibration analysis, electrical circuit modeling, and wave mechanics could benefit from the enhanced computational efficiency and accuracy provided by this method.

In conclusion, this study underscores the utility of Bernoulli polynomials not only as a theoretical tool but also as a practical and efficient method for addressing real-world problems in engineering and physics. Their distinct advantages, such as ease of programming, computational efficiency, and adaptability to diverse problems, make them a promising alternative to traditional numerical methods. Future research could further explore these applications, solidifying Bernoulli polynomials as a cornerstone technique in applied mathematics and engineering problem-solving.

Author Statement

The author is solely responsible for the conceptualization, methodology, data collection, analysis, and manuscript preparation.

Conflict of Interest

The author declares no conflict of interest.

References

Apostol, T. M. (1976). Introduction to Analytic Number Theory. *Springer-Verlag*, Newyork.

Bahşi, M., Çevik, M., & Sezer, M. (2018). Jacobi polynomial Solutions of Volterra Integro-Differential Equations with Weakly Singular Kernel. *New Trends in Mathematical Sciences, 6*(3), 24-38.

Baykuş, N., & Sezer, M. (2017). Hybrid Taylor-Lucas Collocation Method for Numerical Solution of High-Order Pantograph Type Delay Differential Equations with Variables Delays. *Applied Mathematics and Information Sciences, 11*(6), 1795-1801.

Biçer, K.E., & Dağ, H.G. (2023). Numerical Solutions of Differential Equations Having Cubic Nonlinearity Using Boole Collocation Method. *Turkish Journal of Mathematics, 47*, 732-745.

Çayan, S., Özhan, B.B., & Sezer, M., (2022). Collocation Approaches to The Mathematical Model of an Euler–Bernoulli Beam Vibrations. *Mathematics and Computers in Simulation, 197*, 32-44.

Çevik, M., Bahşi, M.M., & Sezer, M. (2014). Solution of The Delayed Single Degree of Freedom System Equation by Exponential Matrix Method. *Applied Mathematics and Computation,* 242, 444-453.

Erdem Biçer, K., & Sezer, M. (2019). A Computational Method for Solving Differential Equations with Quadratic Nonlinearity by Using Bernoulli Polynomials. *Thermal Science, 23*(1), 275-283.

Erdem, K., & Yalçınbaş, S. (2012a). Bernoulli Polynomial Approach to High-Order Linear Differential-Difference Equations. *AIP Conf. Proc.,* 1479, 360-364.

Erdem, K., & Yalçınbaş, S. (2012b). Numerical Approach of Linear Delay Difference Equations with Variable Coefficients in Terms of Bernoulli Polynomials. *AIP Conf. Proc.,* 1493, 338-344.

Erdem, K., Yalçınbaş, S., & Sezer, M. (2013). A Bernoulli Polynomial Approach with Residual Correction for Solving Mixed Linear Fredholm Integro-Differential-Difference Equations. *Journal of Difference Equations and Applications*, 19, 1619-1631.

Gülsu, M., Yalman, H., & Sezer, M. (2011). A New Hermite Collocation Method for Solving Differential Difference Equations. *Applications and Applied Mathematics: An International Journal (AAM),* 6(1), 116-129.

Inman, D.J. (2001). Engineering Vibration, second ed. *Prentice-Hall*, New Jersey.

Kurt, N., & Çevik, M. (2008). Polynomial Solution of The Single Degree of Freedom System by Taylor Matrix Method. *Mechanics Research Communications*, *35*(8), 530-536.

Kürkçü, Ö.K., Aslan, E., & Sezer, M. (2016). A Numerical Approach with Error Estimation to Solve General Integro-Differential-Difference Equations Using Dickson Polynomials. *Applied Mathematics and Computation, 276*, 324-339.

Savaşaneril, N. B. (2018). Laguerre Series Solutions of The Delayed Single Degree of Freedom Oscillator Excited by an External Excitation and Controlled by a Control Force. *Journal of Computational and Theoretical Nanoscience*, *15*(2), 606-610.

Savaşaneril, N. B. (2023). Lucas Polynomial Solution of a Single Degree of Freedom System. *Sci. Res. Comm., 3*(1).

Sezer, M., & Kaynak, M. (1996). Chebyshev Polynomial Solutions of Linear Differential Equations. *Int. J. Mat. Educ. Sci. Technol, 27*(4), 607-618.

Yıldız, G., & Sezer, M. (2019). Truncated Bell Series Approach to Solve Systems of Generalized Delay Differential Equations with Variable Coefficients. *Turkish Journal of Mathematics and Computer Science, 11*, 105-113.

Yüzbaşı, Ş., Şahin, N., & Sezer, M. (2012). A Collocation Approach to Solving the Model of Pollution for a System of Lakes. *Mathematical and Computer Modelling, 55*(3-4), 330-341.

Yüzbaşı, Ş., & Karaçayır, M. (2017). An Exponential Galerkin Method for Solutions of HIV Infection Model of CD4+ T-Cells. *Computational Biology and Chemistry*, 67, 205-212.

Yüzbaşı, Ş., & Yıldırım, G. (2021). A Laguerre Approach for Solving of The Systems of Linear Differential Equations and Residual Improvement. *Computational Methods for Differential Equations, 9*(2), 553-576.